MICHAEL LEE-CHIN & FAMILY INSTITUTE
FOR STRATEGIC BUSINESS STUDIES

Beta Uncertainty as a Barrier to Arbitrage and the Impact on Anomaly Returns*

Working Paper Series in Strategic Business Valuation
WP 2024-06

Ronald Balvers
DeGroote School of Business, McMaster University
balvers@mcmaster.ca

Yufeng Han
Belk College of Business, University of North Carolina - Charlotte
yhan15@uncc.edu

Ou Hu
Zanvyl Krieger School of Arts and Sciences, Johns Hopkins University
ohu@jhu.edu

Zhaodan Huang
Department of Business and Economics, Utica University
zhuang@utica.edu

* We are grateful to Alexander Barinov, Sulei Han, and seminar participants at the Financial Management Association Annual Meetings, the Central University of Finance and Economics (CUFE), Wuhan University, and Universita Cattolica del Sacro Cuore.

First Draft: December 2022
Current Version: December 2024

Beta Uncertainty as a Barrier to Arbitrage and the Impact on Anomaly Returns

Abstract

Beta uncertainty creates unavoidable risk in exploiting anomalies, suggesting an unexplored barrier to arbitrage. We measure beta uncertainty from parameter dynamics and estimation risk, decoupling it from idiosyncratic risk in a Bayesian market model accommodating separate processes for beta and idiosyncratic volatility. Anomalies with higher beta uncertainty generate substantially higher returns. For individual stocks, beta uncertainty is found to reduce arbitrage activity directly, thereby enhancing mispricing. These results support the arbitrage hurdle mechanism versus other hypotheses regarding beta uncertainty. The uncovered arbitrage barrier provides new evidence supporting attribution of observed anomalies to mispricing that cannot be fully corrected by arbitrage.

Keywords: Beta uncertainty, anomaly returns, barriers to arbitrage, Bayesian MCMC, stochastic volatility, mispricing

JEL classification: G11, G14.

Authors:
Ronald Balvers, Yufeng Han, Ou Hu, Zhaodan Huang

1 Introduction

In the last several decades, empirical research has documented a vast number of anomalies earning significant abnormal returns. Many anomalies are found to be robust and persistent both statistically and economically (see, e.g., Chen, 2021; Chen and Zimmermann, 2022). To the extent that many anomalies are driven by mispricing (Chu, Hirshleifer, and Ma, 2020; Han, Lu, Xu, and Zhou, 2024), a fundamental question arises: why don't investors, equipped with extensive knowledge of these well-established anomalies, exploit them more aggressively, thereby eliminating the anomalies?

We posit that uncertainty surrounding risk loading, or beta uncertainty, serves as a significant barrier to arbitrage, precluding investors from fully exploiting anomalies. It is widely held that arbitrage barriers, such as idiosyncratic risk, constrain arbitrageur capacity to correct mispricing in equity markets (for instance, Shleifer and Vishny, 1997; Pontiff, 2006; Gromb and Vayanos, 2010; McLean, 2010; Lam and Wei, 2011; Stambaugh, Yu, and Yuan, 2015; Cao and Han, 2016). However, in the case of long-short anomaly portfolios, idiosyncratic volatility tends to be small due to the inclusion of hundreds of stocks on both long and short sides.

Nevertheless, we find a notable degree of remaining uncertainty, resulting from the systematic risk of the strategy – the beta uncertainty associated with the anomaly. This uncertainty surrounding the anomaly beta poses challenges for investors in devising effective trading strategies. The majority of long-short anomaly portfolios are not market neutral, exposing investors to market risk. Transporting alpha – avoiding market exposure in the process of exploiting anomalies – only works to the extent that the expected market exposure is eliminated. With uncertain risk loadings, return realizations retain an unpredictable net exposure in positive or negative direction. Neither is the uncertainty of loadings necessarily reduced in diversified anomaly portfolios, even though this may be the case for idiosyncratic risk.

To evaluate the barriers to arbitrage as an explanation for anomaly persistence, we need to account for both beta uncertainty and idiosyncratic risk. However, it is difficult in practice to separate beta uncertainty from idiosyncratic volatility. The reason is that the standard error of beta in a time series regression of portfolio returns on market returns, as a measure of beta uncertainty, is per definition equal to the idiosyncratic standard deviation of the portfolio return multiplied by a term that does not vary cross-sectionally. Additionally, to obtain a comprehensive gauge of beta uncertainty it is necessary to account not only for parameter estimation risk but also for time variation in betas.

A modeling approach that explicitly accounts for the dynamics of beta and idiosyncratic volatility, enables us to obtain a suitable measure of beta uncertainty and to disentangle beta uncertainty from idiosyncratic risk. We introduce a modified market model with stochastic volatility and stochastic beta, in which beta uncertainty and idiosyncratic risk are driven by separate random processes. To handle the model complexity, we employ a Bayesian Markov Chain Monte Carlo (MCMC) method and a nonlinear particle filtering process in estimating both the conditional volatility of beta ($BVOL$), our proxy for beta uncertainty, and the conditional idiosyncratic volatility ($IVOL$).

We estimate the model for each of the 207 anomaly long-short spread portfolios provided by Chen and Zimmermann (2022). We find that beta varies considerably across anomalies, but that most anomalies have significant non-zero beta, ranging from -0.735 to 1.211. In addition, $BVOL$ is an order of magnitude higher than $IVOL$, which hints at the relative importance of their respective roles as arbitrage barriers and the corresponding effects on anomaly returns.

If beta uncertainty acts as an arbitrage barrier, anomalies characterized by high beta uncertainty are expected to exhibit elevated future returns. We sort anomaly portfolios into ten deciles based on their *beta uncertainty* levels ($BVOL$), and find that the average returns of these portfolios increase substantially across the deciles. The return differential between the top and bottom deciles amounts to 0.689 percent per month with a corresponding t-statistic of 6.72. Even after adjusting for systematic risk with the Fama-French five-factor model, the differential remains similarly high at 0.695 percent per month with a t-statistic of 4.96.

When we examine the long legs and short legs of the anomalies separately, both are significantly affected by beta uncertainty. We observe that the short legs contribute more than twice as much to the return disparity from differences in beta uncertainty as the long legs, which is consistent with beta uncertainty measured by $BVOL$ serving as an effective arbitrage barrier complementing the shortselling costs.[1]

Separating the data by sentiment state allows a close look at the impact of beta uncertainty when mispricing varies, and may shed a new light on how mispricing reacts to arbitrage. We find that the $BVOL$ premium is robust across the sentiment regimes: 0.756 percent (t-stat of 4.18) per month for High Sentiment and 0.647 percent (t-stat of 5.27) for Low Sentiment. In contrast, as sentiment increases from low to high, we find that, for the short legs, the return differential between the top and the bottom $BVOL$ deciles almost doubles from −0.374 to −0.613 percent per month, whereas, for the long legs, the differential halves from 0.274 to 0.142 percent per month. These offsetting results explain why the $BVOL$ premium is robust across the sentiment states and are consistent with Stambaugh, Yu, and Yuan (2012) – high sentiment states are characterized by the prevalence of overpricing, while underpricing is more prevalent in low sentiment states.

The observation that anomaly returns are predictable from beta uncertainty, viewed as a plausible arbitrage barrier, coupled with how its impact varies with mispricing levels, suggests that a new variable, $BVOL$ confirms the familiar, but still contentious, market perspective: anomaly returns exist because traders with particular biases cause price deviations from fundamentals that are imperfectly offset by rational traders. $BVOL$, as an unexplored

[1]The literature has previously documented shortselling costs as an impediment to arbitrage (e.g., Jones and Lamont, 2002; Nagel, 2005; Duan, Hu, and McLean, 2010; Engelberg, McLean, and Pontiff, 2018; Muravyev, Pearson, and Pollet, 2022). Our results here accord with the view of Chu et al. (2020) and others that the short side is more sensitive to arbitrage barriers that make it riskier and costlier to sell short. Chen, Han, and Pan (2021) identify a further interesting barrier to direct arbitrage, arguing that arbitrageurs (hedge funds) do better in exploiting mispricing by forecasting changes in sentiment. Direct arbitrage exposes the investors to sentiment risk (primarily because of shorting overpriced stocks), making market timing based on sentiment a more profitable alternative.

alternative arbitrage barrier, provides a novel lens through which to test this view. It is challenging to conceive how traditional risk premia could generate a similar pattern of empirical results.

To control for the impact of $IVOL$, we run horse races between $BVOL$ and $IVOL$ in Fama-MacBeth cross-sectional regressions of future returns. Notably, both $BVOL$ and $IVOL$ exhibit positive and highly significant coefficients, across the entire sample period as well as during periods characterized by high or low sentiment. To further disentangle the individual impacts on future returns, we orthogonalize $BVOL$ from $IVOL$ by regressing it on $IVOL$ and squared $IVOL$, and use the residuals in the Fama-MacBeth regressions. The orthogonalized $BVOL$ is still positive and highly significant. When we orthogonalize $IVOL$ with respect to $BVOL$, comparable results are obtained. Upon standardizing both variables, we observe that $BVOL$ exerts a more pronounced influence on future returns compared to $IVOL$. Similar results are observed when we double-sort the anomalies based on both variables – the spreads between the top and bottom deciles are consistently larger and more significant for $BVOL$ than for $IVOL$.

We also investigate anomaly performance over longer horizons. McLean and Pontiff (2016) document a decline in anomalies out of sample and note that anomalies exhibiting higher in-sample returns tend to decay faster. They argue that high past returns attract more capital from arbitrageurs and prompt heightened arbitrage activities. Building on this, we propose a positive interaction between beta uncertainty and the arbitrage opportunities captured by past returns. We anticipate that beta uncertainty attenuates the diminishing effect of high past returns on future anomaly returns because the higher risk of arbitrage due to beta uncertainty reduces the attractiveness of arbitrage invoked by the high past returns. Our empirical findings support this hypothesis. While past cumulative returns are significantly negatively related to future cumulative returns over various horizons, the effect of the interaction between $BVOL$ and past returns on future cumulative returns is always significant and positive. In contrast, the effect of the interaction between $IVOL$ and past returns is consistently negative, though not always significant.

We extend the analysis to the individual firm level instead of anomaly portfolios to reconcile our results with previous literature and to provide further evidence and validation. We expect that higher beta uncertainty increases anomaly strength from a micro (firm level) perspective. In Fama-MacBeth regressions we interact beta uncertainty with the mispricing score (MPS) from Stambaugh et al. (2015) but subtracting 50 so that $MPS > (<) 0$ indicates overpricing (underpricing). We examine overpriced stocks and underpriced stocks separately, and as expected, find that the interaction term has a significantly negative coefficient, the same sign as the coefficient of MPS in the sample of overpriced stocks, but not in the sample of underpriced stocks, consistent with the hypothesis that beta uncertainty imposes an arbitrage barrier aggravating the mispricing in already overpriced stocks. In contrast, for $IVOL$ we observe no significant interaction with MPS, either for overpriced or underpriced stocks.

Further, we aim to provide direct evidence that beta uncertainty reduces investor arbitrage activities. In the first test, we follow Hanson and Sunderam (2014) to use firm level monthly short-selling interest as a proxy for the positions of arbitrageurs. In the regressions of short interest on $BVOL$ and other control variables, we find that $BVOL$ indeed has a significantly negative coefficient. However, to our surprise, $IVOL$ has a significantly positive coefficient.[2] In the second test, we employ an alternative measure of arbitrage activity introduced by Lou and Polk (2022), co-momentum. It confirms the results based on short interest that (average) $BVOL$ negatively affects arbitrage activity, as well as confirming the unexpected result that (average) $IVOL$ positively affects arbitrage activity.

Previous literature on beta uncertainty does not relate beta uncertainty to arbitrage attenuation and exclusively focuses on individual stocks. Armstrong, Banerjee, and Corona (2013) theoretically show that a stock's expected return is negatively related to beta un-

[2] Idiosyncratic volatility is a multifaceted variable. We suspect that the heightened short-selling activities for high $IVOL$ stocks could be related to the $IVOL$ puzzle first documented by Ang, Hodrick, Xing, and Zhang (2006). Alternatively, in the Miller (1977) perspective, high $IVOL$ stocks are stocks with more diverse opinions, for which the optimistic views are expressed more easily, which attracts short-seller arbitrage.

certainty because the stock price is a convex function of its beta, and provide supporting empirical evidence using the standard error of the beta estimate as the measure of beta uncertainty. Using different beta uncertainty measures, Hollstein, Prokopczuk, and Wese Simen (2020) similarly find that stocks of higher estimated beta uncertainty significantly underperform those stocks of lower beta uncertainty.

Our paper aligns with the essence of Barahona, Driessen, and Frehen (2021) who argue that factor loading uncertainty reduces arbitrage demand, emphasizing ambiguity aversion. The ambiguity as to the factor loading forecast reduces arbitrageur demand. Our paper is also related to a larger literature on parameter uncertainty, such as Da, Nagel, and Xiu (2023), Lassance, Martín-Utrera, and Simaan (2024), DeMiguel, Martín-Utrera, and Nogales (2015), Bidarkota, Dupoyet, and McCulloch (2009), Garlappi, Uppal, and Wang (2007), Kan and Zhou (2007), and Lewellen and Shanken (2002). Papers in this literature study the implications of parameter uncertainty and investor learning on asset prices and optimal portfolios choice.

Most relevant, in the context of arbitrage impact on anomaly returns, is Da et al. (2023) who consider statistical limits on arbitrage in an APT context with unknown alphas, showing that measured anomaly returns depend on the extent of alpha uncertainty, although these anomaly returns are not feasible for investors in real time without perfect information about the underlying alphas. Lassance and Martin-Utrera (2024) find that the degradation of a parameter signal out of sample decreases the benefit of using arbitrage signals, to the point that incorporating an arbitrage element in an optimal portfolio may only be beneficial during high sentiment periods. Barroso and Detzel (2021) argue in the context of volatility-managed portfolios that arbitrage is too costly, and may be profitable only if transaction costs are mitigated by using stocks that are easy to arbitrage, counter to the prevalent view that arbitrage profitability is concentrated on the stocks that are the hardest to arbitrage.

Our results support the arbitrage hurdle channel versus other channels for the importance of beta uncertainty proposed by Armstrong et al. (2013), Hollstein et al. (2020), and Boloorforoosh, Christoffersen, Fournier, and Gouriéroux (2020). Most importantly, the proposed role of beta uncertainty presents a new empirical angle that may be exploited to provide independent evidence for a particular theory of anomalies. As proposed by DeLong, Shleifer, Summers, and Waldmann (1990), Shleifer and Vishny (1997), Stambaugh et al. (2015), and others, stock prices are affected by non-fundamentals traders, whereas arbitrage by fundamentals traders restores prices partway back to fundamentals. Arbitrage barriers preclude the full reversion to fundamentals, thus leaving predictable anomaly returns. Our identification of beta uncertainty as a previously unexplored arbitrage barrier allows a different look at the factors affecting anomalies, yielding supplemental support for the view that mispricing generates persistent anomalies when arbitrage is impaired.

In contrast to the previous literature, which estimates $BVOL$ and $IVOL$ separately, thus ignoring their high correlation and the resulting compounding effect, our paper is the first to recognize the intertwined nature of the two variables, and formally model and estimate them simultaneously. In our stochastic model, $BVOL$ and $IVOL$ are identified from distinct stochastic processes, breaking the tight link between them. Because $IVOL$ has been found to proxy for many risks and has been linked to numerous firm characteristics, it may be worthwhile to reassess the various effects of $IVOL$ with estimates such as ours.

The remainder of the paper is organized as follows. Section 2 presents the theoretical argument for why beta uncertainty operates as an important arbitrage barrier. Section 3 discusses our stochastic market model and its estimation and prediction. Section 4 discusses the data used. Section 5 presents the main empirical results demonstrating the predictability stemming from beta uncertainty. Section 6 disentangles the effects of $BVOL$ and $IVOL$. Section 7 discusses the influence of $BVOL$ on the persistence of anomalies. Section 8 presents further evidence from firm-level analysis and section 9 concludes.

2 Beta uncertainty as a barrier to arbitrage

To exploit market anomalies, active investors ("arbitrageurs") consider investment in anomaly positions in the context of their total asset portfolio. The marginal contribution of anomaly holdings is most transparent when converted to a market neutral position, enabling "alpha transport". When the market beta (in a more general context, loadings on any set of systematic risk factors) of such anomaly position is uncertain, the arbitrage position necessarily incurs an unknown degree of systematic risk, potentially discouraging arbitrage.

We assess the issue in the context of the basic market model, the Treynor-Black formulation, augmented by relaxing the assumption that market betas are known exactly, and we also introduce stochastic volatilities into the model. The modified market model, presented in terms of excess returns, is

$$
\begin{aligned}
r_{jt} &= \alpha_j + \beta_{jt} r_{mt} + \sigma_{jt} e_{jt}. \\
r_{mt} &= \mu_m + \sigma_{mt} e_{mt,}
\end{aligned}
\tag{1}
$$

Here r_{jt} is the monthly excess return of asset j at month t, with abnormal return α_j, stochastic market beta, β_{jt} and stochastic idiosyncratic volatility σ_{jt}. The monthly market excess return at month t is r_{mt} with constant mean return μ_m and stochastic volatility, σ_{mt}. Details of the model for estimation purposes, considering in particular the dynamics of the uncertainty and stochastic variation, are in the next section. The specific dynamics of stochastic volatility helps distinguish and identify beta uncertainty from idiosyncratic variance. What is relevant conceptually, for now, is that β_{jt} is stochastic, and all the random processes are indpendent.

Arbitrageurs invest in an anomaly by taking a zero-investment position in a set of assets with non-zero alpha, and then transport the alphas by creating expected market-neutral positions,

$$r_{at} = \alpha_a + (\beta_{at} - \hat{\beta}_{at-1}) r_{mt} + \sigma_{at} e_{at,} \tag{2}$$

where $\hat{\beta}_{at-1} = E_{t-1}\beta_{at}$, the conditional expectation of the position's market beta used to attain as close as possible a market-neutral position. Overall, we assume that an arbitrageur invests in anomaly a with weight w and with total investment in the market portfolio (including the market investment needed to eliminate expected market sensitivity of the active portfolio) normalized to one.

The total return of the strategy is

$$r_t = w_{t-1}r_{at} + r_{mt} \tag{3}$$

Both the active anomaly position and the passive market position are zero investment so there is no need to further normalize the portfolio weights. The squared Sharpe ratio, $SR^2_{t-1} = \frac{E_{t-1}(r_t)^2}{Var_{t-1}(r_t)}$, generated by the strategy is

$$SR^2_{t-1} = \frac{(w_{t-1}\alpha_a + \mu_m)^2}{w^2_{t-1}Var_{t-1}(r_{at}) + Var_{t-1}(r_{mt})}, \tag{4}$$

where Var_{t-1} denotes conditional variance. Note that the covariance between r_{at} and r_{mt} that would show up in the denominator is by design equal to zero. Arbitrageurs choose the weight on the anomaly position to maximize the (squared) portfolio Sharpe Ratio. In equation (4), $Var_{t-1}(r_{at}) = \sigma^2_{at-1} + Var_{t-1}[(\beta_{at} - \hat{\beta}_{at-1})r_{mt}]$, and the multiplicative variance in the last term equal to $Var_{t-1}[(\beta_{at} - \beta_{at-1})r_{mt}] = \sigma^2_{\beta_a t-1}(\sigma^2_{mt-1} + \mu^2_m)$, given that β_{at} and r_{mt} are independent.[3] The optimal weight from the first-order condition then becomes

$$w^*_{t-1} = \frac{\alpha_a/\mu_m}{\sigma^2_{\beta_a t-1}(1 + SR^2_{mt-1}) + (\sigma^2_{at-1}/\sigma^2_{mt-1})}. \tag{5}$$

From the solution in equation (5), the optimal weight, w^*_{t-1}, decreases in the conditional volatility of beta, $\sigma_{\beta_{at-1}}$: the higher the beta uncertainty, the smaller the weight arbitrageurs put on the active anomaly portfolio. Higher beta uncertainty induces arbitrageurs to allocate

[3]For independent random variables, x and y, we employ $Var(xy) = Var(x)Var(y) + Var(x)(Ey)^2 + Var(y)(Ex)^2$.

less investment to the anomaly. Arbitrage is slowed by high beta uncertainty. The conditional idiosyncratic volatility σ^2_{at-1} is negatively related to the optimal active investment weight in equation (5), $\partial w^*_{t-1}/\partial\sigma^2_{at-1} < 0$, implying that, in addition to beta uncertainty, idiosyncratic risk slows down arbitrage as well, consistent with the common belief that idiosyncratic volatility is an arbitrage risk.

To contrast the impact of beta uncertainty and idiosyncratic uncertainty, it is useful to bear in mind the positions s_{it-1} in the underlying individual zero-investment asset positions i that together generate the anomaly return $r_{at} = \sum_{i=1}^{n} s_{it-1} r_{it}$. Assume for simplicity the special case that, for anomaly a, each of the n separate positions have identical alphas α, identical idiosyncratic risk σ_e without correlation, identical beta uncertainty σ_β, and identical correlation of beta risk among the positions ρ. All are also constant over time. These assumptions represent the presumption that correlation in idiosyncratic risk among the asset positions is negligible due to diversification and that beta risk among the asset positions is more highly correlated as, for instance, some non-market characteristics have a common component. The assumptions also imply that optimal arbitrage investment in each position s_i will be identical and normalized to $1/n$, which lines up with the data in which anomaly returns are constructed as equal-weighted averages of the asset position returns.

Under these simplifying assumption we have that

$$\begin{aligned} \sigma^2_{at-1} &= Var\left(\sum_{j=1}^{n} e_{jt}/n\right) = \frac{\sigma^2_e}{n}, \\ \sigma^2_{\beta_a t-1} &= Var\left(\sum_{j=1}^{n} \beta_{jt}/n\right) = [1+(n-1)\rho]\frac{\sigma^2_\beta}{n} \end{aligned} \tag{6}$$

First assume that $\sigma^2_\beta = 0$. Without beta uncertainty, the result in Pontiff (2006) holds, substituting the appropriate values from equation (6) into equation (5). The price pressure to neutralize anomalies resulting from arbitrage demand for specific assets does not depend on how many asset positions can be utilized to exploit the anomaly: the arbitrage investment $w^*_{t-1} \times s_{it-1}$ in asset position i does not change with n the number of exploitable assets.

If we now allow for beta uncertainty, $\sigma_\beta^2 > 0$, but ignore correlation of beta uncertainty, this uncertainty is magnified by $1 + SR_{mt-1}^2$ in equation (5). The relevance of the conditional market Sharpe ratio results from the multiplicative interaction of beta uncertainty with the market factor uncertainty.

If we also admit positive correlation among asset betas, $\rho > 0$, this further increases the importance of beta uncertainty in mitigating arbitrage, in absolute terms as well as relative to the idiosyncratic risk. In the limit, as the number of exploitable asset positions becomes infinite, the impact on individual asset positions goes to zero. Arbitrage then has no impact on individual assets, even though, in sum, the total weight of the arbitrage investment $\sum_{i=1}^{n} w_{t-1}^* \times s_{it-1} = w_{t-1}^*$ directed to anomaly a remains strictly positive, converging to $\frac{\alpha/\mu_m}{\rho(1+SR_m^2)\sigma_\beta^2}$. As we consider empirically both the overall anomaly positions and the individual asset positions, these distinctions are useful to keep in mind.

To get an idea of the quantitative importance of the beta volatility channel for impeding anomaly arbitrage, consider reasonable numerical values for the arbitrage portfolio weights in equation (5). It makes sense when we consider transaction costs associated with the anomalies to consider a holding period in the ballpark of one month. Accordingly, we will consider representative numerical values at the monthly frequency.

From Panel B of Table 1, we take the average anomaly alpha as a representative anomaly return so that $\alpha_a = 0.453$. Based on historical market excess returns (using the one-month t-bill rate as the risk-free return) since 1926 we have that the monthly market excess return is $\mu_m = 0.685$ percent and the associated market excess return volatility is $\sigma_m = 0.053$. If we ignore beta volatility ($\sigma_{\beta_a} = 0$) then, and the average anomaly idiosyncratic volatility from Table 1 as $\sigma_a = 0.027$. Then $\sigma_a^2/\sigma_m^2 = 0.260$. In this case the pressure on the anomaly caused by arbitrage is represented by $w^* = 2.54$ for every 1 unit held in the market excess return. If we set beta volatility to the anomaly average of 0.220 from Panel B of Table 1 then $\sigma_{\beta_a}^2 = 0.05$ and $w^* = 2.14$. Not all that much smaller. However, the beta volatility varies significantly across anomalies. The average level of beta volatility for the top decile of anomalies equals 0.465 from Table 1, then we have $\sigma^2_{\beta_a} = 0.216$ and $w^* = 1.38$, implying substantially less arbitrage pressure, all else equal, for the high beta volatility anomaly with 1.38 "price pressure" relative to the average anomaly for which we found a price pressure number of 2.14.

If, just to get an idea of magnitudes, we posit that in "equilibrium" the price pressure for each anomaly would be the same, and equal to that for the average case at 2.14 then the lower price pressure for the high beta volatility case, supposedly leading to less arbitrage and, hence, higher remaining anomaly returns, keeping all else equal (in particular assuming equal idiosyncratic volatility for simplicity) we find that with alpha equal to 0.453 for the average case, it would need to be (2.14/1.38) * 0.453 = 0.703 percent per month for the high beta volatility anomalies. This implied alpha for high beta volatility anomalies is not far from what we find in our formal empirical analysis, 0.744 percent per month.

3 Model of anomaly returns

3.1 Motivation

The market model described in the previous section will be fully specified for estimation purposes. The dynamics of factor sensitivities, along with their volatilities and the volatilities of the idiosyncratic and market shocks, are explicitly defined to allow beta uncertainty and idiosyncratic volatility to be estimated from distinct processes. While this approach is more complex than conventional methods that estimate beta uncertainty ($BVOL$) and idiosyncratic risk ($IVOL$) separately, it is essential for capturing the intertwined nature of these two variables.

Consider the common estimates of $BVOL$ and $IVOL$ obtained from the basic market model,

$$r_{it} = \alpha_i + \beta_i r_{mt} + e_{it}, \tag{7}$$

where r_{it} is the monthly excess return of portfolio i for month t, r_{mt} is the monthly market excess return. We may conventionally estimate the coefficients using monthly returns in T-month rolling windows (usually $T = 60$), rolling over one month at a time to obtain monthly estimates of β_i. The $BVOL$ estimate is then the standard error of β_i. This method for estimating $BVOL$ is employed by Armstrong et al. (2013). The standard error of beta, used as the $BVOL$ measure, equals

$$\sigma(\hat{\beta}_i) = \frac{\sigma(\hat{e}_i)}{\sqrt{\sum_{t=1}^{T}(r_{mt} - \bar{r}_m)^2}}, \tag{8}$$

We now obtain $IVOL$ straightforwardly as the standard deviation of the residual, $\sigma(\hat{e}_i)$ from the same regression. It follows that $BVOL$ is directly related to $\sigma(\hat{e}_i)$ and, therefore, in the cross-section is mechanically perfectly correlated with $IVOL$.

Probably for this reason, Armstrong et al. (2013) estimate $IVOL$ in a different way which is also the standard approach for estimating idiosyncratic volatility. Following Ang et al.'s (2006)'s findings, a large literature has focused on the puzzling relation between $IVOL$ and future stocks returns. In this literature, idiosyncratic volatility ($IVOL$) is estimated from daily returns, and the most common approach is to use one month of daily returns (see, e.g., Ang, Hodrick, Xing, and Zhang, 2009; Han and Lesmond, 2011; Han, Hu, and Lesmond, 2015), which is the approach that Armstrong et al. (2013) use for $IVOL$ estimation.

Accordingly, the estimation method for $IVOL$ is identical to our initial description, in which it is the standard deviation of the residual, $\sigma(\hat{e}_i)$ from the market model regression, except that instead of using five years of monthly data, it uses one month of daily data. To generate meaningful empirical differences between $BVOL$ and $IVOL$, Armstrong et al. (2013) mix estimation frequencies and intervals.

While it is common to estimate betas with 60 monthly data points (although 12 months of daily data, when daily data are available, is more typical) and idiosyncratic risk from one month of daily data, there is no fundamental reason for these frequencies. Differences in frequency generate differences in BVOL and IVOL that are hard to interpret and essentially meaningless from the perspective of identifying true differences in the underlying variables. In the case of Armstrong et al. (2013), it is not clear how serious the issue is since presumably they use the variables as proxies for unrelated issues – $BVOL$ as an indicator of convexity impact and $IVOL$ to capture all barriers to arbitrage. In our case, however, both capture barriers to arbitrage and it is vital to distinguish them correctly.

To properly characterize investor uncertainty about market loadings, it is essential to include the dynamic element. The conventional $BVOL$ proxy is implicitly based on constant betas and omits a key element of beta uncertainty. Market participants are uncertain about factor loadings not only because of estimation risk associated with a constant parameter but also because of time variation in the latent parameter that cannot be fully inferred from past observations. Our model should explicitly account for this additional component of $BVOL$.

Omitting a key component of $BVOL$ has implications for $IVOL$ because $BVOL$ and $IVOL$ are intertwined. Variability missed in the one may be captured by the other. To understand this in an intuitive way, consider returns generated from a simple model (a special case of our empirical model) with stochastic beta,

$$\begin{aligned} r_{it} &= \alpha_i + \beta_{it} r_{mt} + e_{it}, \\ \beta_{it} &= \beta_i + \eta_{it} \end{aligned} \tag{9}$$

If one incorrectly estimates a simple model with a constant beta as in equation (7), the unexplained variation, $e_{it} + \eta_{it} r_{mt}$, used to estimate idiosyncratic volatility, contains a component of beta uncertainty. Therefore, when the coefficient on $IVOL$ is significant, it could be the beta uncertainty component that makes it so. This highlights the necessity to have models that explicitly accommodate the dynamics of these two components needed to estimate them simultaneously. Our model presented in the following subsection provides a reasonable design for this purpose.

3.2 Model specification

To capture beta uncertainty and separate beta uncertainty from idiosyncratic volatility, we provide further details to the modified market model introduced in the previous section.[4] Restating first the modified market model for anomaly returns from the previous section,

$$
\begin{aligned}
r_{jt} &= \alpha_j + \beta_{jt} r_{mt} + \sigma_{jt} e_{jt}, \\
r_{mt} &= \mu_m + \sigma_{mt} e_{mt},
\end{aligned}
\tag{10}
$$

where r_{jt} is the monthly excess return of anomaly j at month t, with market-risk adjusted abnormal return α_j, stochastic market beta, β_{jt}, and stochastic volatility, σ_{jt}, and denoting r_{mt} the monthly market excess return at month t with mean return μ_m and stochastic volatility, σ_{mt}.

We now specify the dynamics of the stochastic components. The anomaly beta and the log-variances of the anomaly return and the excess market return all follow autoregressive AR(1) processes. Let h_{jt} denote the log variance of the anomaly return j at time t, $h_{jt} = ln(\sigma_{jt}^2)$; similarly, let $h_{mt} = ln(\sigma_{mt}^2)$, be the log variance of the market excess return at time t. We have

$$
\begin{aligned}
\beta_{jt} &= \phi_{j0} + \phi_{j1}(\beta_{jt-1} - \phi_{j0}) + \phi_{j2}\eta_{jt}, \\
h_{jt} &= a_{j0} + a_{j1}(h_{jt-1} - a_{0j}) + a_{j2}\epsilon_{jt}, \\
h_{mt} &= b_{m0} + b_{m1}(h_{mt-1} - b_{m0}) + b_{m2}\epsilon_{mt},
\end{aligned}
\tag{11}
$$

The error terms are assumed to be distributed as a multivariate standard normal, i.e., $\mathbf{N}(0, \mathbf{I})$, where $\mathbf{I}$ is the identity matrix. For simplicity, we assume that the market risk premium μ_m is constant, which does not affect the dynamics of the anomaly return. The set of all parameters in the model is then denoted as $\theta = (\alpha_j, \beta_{jt}, \mu_m, \phi_{j0}, \phi_{j1}, \phi_{j2}, a_{j0}, a_{j1}, a_{j2}, b_{m0}, b_{m1}, b_{m2})$.

[4] Although empirically relevant we abstract from shortselling costs and time-varying mispricing regimes to keep the model tractable.

Beta uncertainty has two dimensions in this model. One is associated with parameter estimation (see, e.g., Lewellen and Shanken, 2002; Kan and Zhou, 2007). This is the uncertainty discussed by Armstrong et al. (2013) in the context of individual stocks. Bayesian models with priors naturally capture this uncertainty. The other dimension, arguably more important, is the uncertainty associated with the dynamics of beta. We model beta as a latent AR(1) process with a distinct random noise. An AR(1) process is simple to estimate but general enough to capture higher order autocorrelations. A major motivation for the model is to separate beta uncertainty from the idiosyncratic volatility of returns. The AR(1) dynamics allows the beta to have a completely separate randomness (uncertainty).

We model idiosyncratic volatility as a stochastic process for two key reasons. First, it accommodates time-varying volatility, a prominent feature in returns often addressed by various GARCH models. However, GARCH models do not allow for separate randomness in the volatility process. Therefore, the second reason is to decouple beta uncertainty from idiosyncratic volatility. This is achieved in the model by using an AR(1) process for beta and a log AR(1) process for idiosyncratic volatility.

3.3 Estimation

We estimate the modified market model for each anomaly with the Bayesian Markov Chain Monte Carlo (MCMC) methods, proposed by Kim, Shephard, and Chib (1998), refined by Chib, Nardari, and Shephard (2002), and extended by Chib, Nardari, and Shephard (2006) and Han (2006) to higher dimensional settings and by Omori, Chib, Shephard, and Nakajima (2007) to accommodate a leverage effect.

In general, MCMC is a simulation-based method designed to sample densities that do not have closed forms. The method generates sample draws from the posterior distribution of parameters by a recursive Monte Carlo sampling process: the transition kernel of a Markov process is constructed such that its limiting invariant distribution is the posterior distribution; the Markov chain is then iterated a large number of times in a Monte Carlo simulation.

After a burn-in period, the Markov chain converges, and the sampled draws are collected as variates from the posterior distribution.

Stochastic volatility models are nonlinear state-space models, which tend to be difficult to estimate. Following Chib et al. (2002), we augment the posterior distribution of parameters to include the latent beta (β_{jt}) and log variances (h_{jt} and h_{mt}), and linearize the model by approximating the distribution of the log of a standard chi-squared random variable with a seven-component mixture of normal distributions. Instead of estimating nonlinear state-space models, we may now work with linear state-space models, which can be estimated efficiently.[5]

3.4 Prediction

After estimation of the model, we can conditionally forecast the beta, $\beta_{t+1,}$ and idiosyncratic volatility, $\sigma_{at+1} = e^{h_{t+1}/2}$, of the anomaly returns through the predictive distribution, $\pi(h_{t+1}, \beta_{t+1}|\mathscr{F}_t, \theta)$, where θ is fixed at the posterior mean. As the predictive distributions of β_{t+1} and h_{t+1} are not available in closed form, we instead generate sample draws from the predictive distribution by the method of composition as below,

$$\pi(h_{t+1}, \beta_{t+1}|\mathscr{F}_t, \theta) = \int \pi(h_{t+1}, \beta_{t+1}|h_t, \beta_t, \mathscr{F}_t, \theta)\pi(h_t, \beta_t|\mathscr{F}_t, \theta)dh_t d\beta_t, \tag{12}$$

where $\pi(h_{t+1}, \beta_{t+1}|h_t, \beta_t, \mathscr{F}_t, \theta)$ is the joint AR(1) transition distribution, and $\pi(h_t, \beta_t|\mathscr{F}_t, \theta)$ is the filtering distribution. Given the sample draws $h_t^{(i)}$ and $\beta_t^{(i)}$ from the filtering distribution, the predictive draws $h_{t+1}^{(i)}$ and $\beta_{t+1}^{(i)}$ are obtained by directly sampling from their transition distributions. The filtering draws can be generated via Bayes' theorem,

$$\pi(h_t, \beta_t|\mathscr{F}_t, \theta) \propto f(r_t|h_t, \beta_t, \theta)\pi(h_t, \beta_t|\mathscr{F}_{t-1}, \theta). \tag{13}$$

[5]For details of the estimation, see Chib et al. (2002) and the other papers cited above.

Therefore, the predictive and filtering distributions are updated sequentially. The prior belief about the latent beta and log variances at time t is the predictive distribution at time $t-1$. The prior is then updated to form the filtering distribution by incorporating the new information about returns at time t. The filtering distribution in turn is used to form the predictive distribution at time t, which is again the prior at time $t+1$, and the updating process repeats again for $t+1$.

Because the stochastic volatility model is nonlinear, the commonly used linear Kalman filter cannot be applied; thus, we employ a nonlinear filter, the auxiliary particle filter, proposed by Pitt and Shephard (1999). Given the filtering particles of h_{t-1} and β_{t-1} conditioned on the information at time $t-1$, the particle filter produces filtering particles of h_t and β_t conditioned on the information at time t, by recursively applying equations (12) and (13).

Once we obtain the sample draws, $\tilde{h}_{t+1}^{(i)}$ and $\tilde{\beta}_{t+1}^{(i)}$ from the predictive distributions, we can compute the beta, the conditional volatility of beta, and the idiosyncratic volatility for the anomaly returns as follows.

$$
\begin{aligned}
\hat{\beta}_{at+1} &= \frac{1}{M}\sum_{i=1}^{M}\tilde{\beta}_{t+1}^{(i)}, \\
\hat{\sigma}_{\beta_a t+1} &= \sqrt{\frac{1}{M-1}\sum_{i=1}^{M}(\tilde{\beta}_{t+1}^{(i)} - \tilde{\beta}_{t+1})^2}, \\
\hat{\sigma}_{e_a t+1} &= \frac{1}{M}\sum_{i=1}^{M} e^{\tilde{h}_{t+1}^{(i)}/2}.
\end{aligned} \tag{14}
$$

In the empirical analysis we take $\hat{\sigma}_{\beta_a t+1}$ as our measure of beta uncertainty, $BVOL$ and $\hat{\sigma}_{e_a t+1}$ as our measure of idiosyncratic volatility, $IVOL$.

While different anomalies have different starting time, for each anomaly, we use all available observations to estimate the model recursively with expanding windows. In each estimation, we generate predictive sample draws for the next five years (60 observation), and enlarge the next estimation window by the same five years. The initial estimation window is 240 observations plus the remainder of the total number of observation divided by the increment (60) so that the final forecast is at the end of the sample period. For example, anomaly AM has a total of 858 monthly observations from July 1951 to December 2022. The remainder of 858 to 60 is 18; thus the initial estimation window is the first 258 observations.

4 Anomaly betas and their volatilities

4.1 Data

Our anomaly data consist of monthly returns on 207 long-short anomaly portfolios, compiled by Chen and Zimmermann (2022) based on the methodologies in the original publications. The data is publicly available at both the monthly and daily frequencies.[6] The data range from January 1926 to December 2022, but many anomalies start at a later time (many accounting anomalies only start from July 1951 or 1952, for instance). Additionally, we use at least the first 20 years to estimate the model. Therefore, our anomaly sample effectively starts from February 1948.

To show how beta uncertainty affects anomaly returns at the individual firm level, we obtain individual stock returns and prices from the Center for Research in Security Prices (CRSP) and firm characteristics, such as book-to-market ratios, from Compustat North America. We include all common stock on the NYSE, AMEX, and NASDAQ. We construct the liquidity measure (AMD) from Amihud (2002) by averaging the daily ratio of the absolute stock return to the dollar trading volume within the month: $AMD_{it} = Avg\left[\frac{|R_{id}|}{\$VOL_{id}}\right]$. Finally, we estimate the beta uncertainty and idiosyncratic volatility for individual firms following equation (10), but require an initial estimation period of 50 years, starting the effective sample in January 1976, because the much higher volatility in individual stock returns requires a longer sample to estimate.

[6]The website is https://www.openassetpricing.com/data/ maintained by Andrew Chen. The anomaly returns we use are those duplicating the original publication papers. Therefore, some may be value weighted and some equal weighted, depending on the original method used in the publication identifying the anomaly. Andrew Chen also makes available the anomaly returns using either value-weighted or equal-weighted format for all anomalies but, in these cases, the number of available anomalies drops to 172

4.2 Beta distribution across 207 anomalies

Since anomaly portfolios are based on long-short zero investment strategies, it is generally believed that these strategies should have near zero exposure to market risk (Lochistoer and Tetlock, 2020). If so, one might assume beta uncertainty to have minimal impact. We argue that beta uncertainty could affect trading in a significant way for two reasons: first, the assumption of near zero exposure to market risk for an anomaly strategy may be invalid; second, even if an anomaly strategy has near zero average exposure, it would still be challenged by the time-varying nature of beta. In either case, beta uncertainty is likely to play an important role.

In this subsection, we show that, even from an average perspective, market betas for most anomalies are non-zero. We estimate the time-series average of the market beta and its t-value for each anomaly and examine its significance. The results, with equal-weighted anomaly averages based on our Bayesian Markov Chain Monte Carlo estimation, are shown in Panel A of Table 1.

Of the 207 anomalies, 56 have a time-series t-statistic larger than 2 with an average β of 0.145, and 144 have a time-series t-statistic smaller than -2 with an average β of -0.157. Only 7 anomalies have insignificant β. The non-zero market risk exposure implies that investors are exposed to market risk while implementing a long-short strategy if these anomalies are exploited without additional market positions.

Even for the remaining few anomalies which do not have an average beta statistically different from zero (an absolute t-value less than 2), investors may still be challenged by not knowing the actual anomaly beta. The beta uncertainty may also leads to arbitrage difficulty for these anomalies. In the next subsection, we further examine the summary statistics of the market beta and, more importantly, the volatility of beta across the anomalies.

4.3 Summary statistics

Panel B of Table 1 presents the summary statistics of the relevant variables. These summary statistics are estimated across the 207 anomalies from the time-series averages for each anomaly. The average monthly return of the anomalies is 0.453% per month, with a median of 0.375% per month. However, some anomalies yield negative time-series average returns as low as -0.488% per month; other anomalies yield positive time-series average returns as high as 4.120% per month. The cross-sectional average (median) of the market β is only -0.070 (-0.052), but there is a large variation across anomalies as indicated by a large standard deviation of 0.219. The smallest time-series average beta for an anomaly is -0.735, while the largest time-series average beta for an anomaly is 1.211. This is consistent with Panel A, which shows that most anomalies have significant market exposure.

Across the anomalies, the average (median) beta volatility ($BVOL$) is 0.220, with a standard deviation of similar magnitude (0.186). While some anomalies have very small and stable $BVOL$, as low as 0.020, others have $BVOL$ as high as 0.982. This supports our argument that even for an anomaly with insignificant beta the volatility in beta can still be sizeable.

Panel B also reports the average idiosyncratic volatility ($IVOL$) across the anomalies, which is only 0.027, small in comparison to the average $BVOL$. The time-series average of $IVOL$ can be as low as 0.007 for some anomalies, and as high as 0.123 for other anomalies. That $BVOL$ is larger than $IVOL$ for anomalies is to be expected because each anomaly portfolio contains hundreds of stocks, so that much of the idiosyncratic volatility is diversified away. However, just as the anomaly market beta level is small but not zero for most anomalies, $IVOL$ is not fully diversified away for many anomalies either. This suggests that, for anomaly portfolios, or for arbitrageurs who trade the anomalies using similar strategies, $BVOL$ instead of $IVOL$ is a more important consideration as an arbitrage barrier.[7]

[7]The diversification impact in anomaly positions for $IVOL$ compared to $BVOL$ can be illustrated by considering the anomaly averages of $IVOL$ and $BVOL$ in relation to their values for individual stocks. The anomaly $IVOL$ average of 0.027 (in Table 1, Panel B) equals that of a stock in the (lowest) 1st percentile of

5 Beta uncertainty and expected returns

If substantial beta uncertainty prevents arbitrageurs from forming effective trading strategies, then it is reasonable to conjecture that the degree of beta uncertainty attached to the anomalies affects future anomaly returns, as follows from equation (5). Ceteris paribus, we expect anomalies of higher (lower) beta uncertainty to yield higher (lower) future returns. We first show that the beta *levels* of anomalies are not associated with future returns whereas the beta *uncertainty* as measured by $BVOL$ directly affects anomaly future returns. We then provide evidence that the return predictability due to beta uncertainty largely stems from the short leg of the anomalies as is expected from Jones and Lamont (2002) and others. In the last part of this section, we provide corroborating parametric evidence using Fama-MacBeth regressions.

5.1 Decile portfolios formed on market beta and $BVOL$

We first conduct a portfolio analysis of the predictive power of the market betas on the returns of the anomalies. At the end of month t, we form decile portfolios based on the rankings of the market betas of the 207 anomalies. Then, the equal-weighted portfolio returns for month $t+1$ are calculated. The third column of Table 2 reports the sample average returns of the decile portfolios sorted by beta from low to high. As can be observed, there is no clear pattern in return across the decile portfolios. The return differential between the 10^{th} and 1^{st} portfolios is about 0.209% per month, which is statistically insignificant. The return after adjustment for the Fama-French five factors is positive but insignificant. This result is consistent with the established empirical consensus that, unconditionally, the security market line is flat.

Subsequently we conduct a similar portfolio analysis considering the effect of the market beta uncertainty ($BVOL$) on the returns of anomalies. At the end of month t, we sort the

the $IVOL$ distribution of all stocks in our sample. In contrast, the anomaly $BVOL$ average of 0.220 equals that of a stock in the 20th percentile of the $BVOL$ distribution of all stocks in our sample.

anomalies by $BVOL$ and form equal-weighted decile portfolios. The second column of Table 2 reports the sample average returns of these decile portfolios. There is a general positive relation between beta uncertainty ($BVOL$) and portfolio returns. The return differential between the top and bottom decile portfolios is 0.689% per month with a t-statistic of 6.72, statistically and economically significant. The abnormal return is similarly economically and statistically significant at 0.695% per month with t-statistic of 4.96, after risk adjustment with the Fama-French five-factor model.

Figure 1 illustrates the significance of sorting the anomalies by $BVOL$. It shows the cumulative abnormal returns from investing in the top quintile of the anomalies with the highest $BVOL$ in comparison to the bottom quintile and the three middle quintiles, with returns adjusted for the five Fama-French risk factors. For the highest quintile of $BVOL$ (representing the anomalies with the largest arbitrage barriers resulting from uncertainty about the systematic risk), the cumulative abnormal anomaly returns reach as high as 1080.4% in the period from July 1968 through December 2022. In contrast, for the bottom quintile the cumulative returns are -45.7%, whereas for the middle quintiles the cumulative returns are 51.2%. Evidently, the anomalies only persist for the highest $BVOL$ quintile. It appears that a high level of beta uncertainty sufficiently impedes arbitrage to preserve initial systematic pricing errors.

Since only the highest $BVOL$ anomalies generate persistent positive returns, it is interesting to see which type of anomalies this applies to. Considering the anomalies in the top $BVOL$ decile, first keep in mind that the $BVOL$ values and rankings change over time, so we select the decile of anomalies with the highest time-series average $BVOL$. What stands out is that 9 out of the 20 anomalies in this decile are based on momentum. These anomalies rely a lot on small growth stocks and also change portfolio composition relatively quickly as momentum of individual stocks changes over time. Each of these aspects may contribute to high beta uncertainty. Irrespective of the cause of high $BVOL$, our theory argues that the momentum anomalies are so robust and persistent because arbitrage activity is hampered by the unavoidable and uncertain market risk exposure inherent in the momentum strategies.

5.2 Beta uncertainty in high versus low sentiment periods

Since every anomaly is a zero-investment strategy, combining a long and a short position, we may examine separately the long legs and the short legs of the decile portfolios and identify the contribution from each side. In Table 3, Panel A shows that the return differential between the tenth and first decile portfolios is -0.465% with a t-stat of -7.45 for the short legs, while that of the long legs is 0.224% with a t-stat of 3.72. The short side, accordingly, contributes substantially more to the return differential than the long side, consistent with the common perception (see, for instance, Stambaugh et al. 2012, 2015; Chu et al. 2020) that the short side is more sensitive to arbitrage barriers which makes it more costly and riskier to sell short.

We use the sentiment index from Baker and Wurgler (2006) to divide the full sample period into high sentiment periods and low sentiment periods, and separately examine the performance of the decile portfolios of the long-short anomaly portfolios as well as the separate long and short legs. The differences in sentiment facilitate a look at the interaction between arbitrage barriers from beta uncertainty and mispricing. Panels B and C of Table 3 provide several results of note.

First, in Table 3 the monthly average returns of all long and short positions in High Sentiment periods are considerably lower than those in Low sentiment periods, consistent with Baker and Wurgler (2006) who show that returns of stocks whose valuations are subjective and difficult to arbitrage are relatively high when sentiment is low, and relatively low when sentiment is high. Baker, Wurgler, and Yuan (2012) further find that high investor sentiment predicts low future returns (and low sentiment predicts high future returns) in global stock markets.

Second, to digest the variation in anomaly returns for different beta uncertainty levels and sentiment regimes, consider the anomaly return for the top decile of $BVOL$ which equals

1.160% for High Sentiment and 0.827% for Low Sentiment, and contrast it with the anomaly return for the bottom decile of $BVOL$ which equals 0.404% for High Sentiment and 0.179% for Low Sentiment. Even though general return levels vary with sentiment, these numbers reveal that the beta uncertainty premium is robust and quite similar across sentiment regimes at 0.756% (t-stat = 4.18) in High Sentiment periods and 0.647% (t-stat = 5.27) in Low Sentiment periods, in both cases consistent with $\partial w^*_{t-1}/\sigma^2_{\beta t-1} < 0$ in equation (5).

Third, the long and short legs returns of the anomaly portfolios, nevertheless, vary under different levels of sentiment. The short leg relative to the long leg contributes about four times as much to anomaly returns, 0.613% versus 0.142%, in High Sentiment periods, but barely more, 0.374 versus 0.274, in Low Sentiment periods. These results are in line with Stambaugh et al. (2012) and Stambaugh et al. (2015). During periods of high sentiment, stocks are more likely overpriced. Therefore, we should observe stronger results for the short legs and weaker results for the long legs because far fewer stocks are underpriced. In contrast, during periods of low sentiment, stocks are more likely underpriced, causing weaker results for the short legs and stronger results for the long legs. These opposite changes in the return differential from the short leg and the long leg result in the similar return differential observed for the long-short portfolio in both High Sentiment and Low Sentiment periods.

The patterns remain consistent whether or not we adjust for risk using the Fama-French five factors, indicating that non-market systematic risk plays a limited role in explaining the results. The specific variation in the importance of beta uncertainty across different sentiment regimes and long versus short positions makes it difficult to imagine alternative explanations in which beta uncertainty matters as an indicator of systematic risk rather than as a barrier to arbitrage.

5.3 Fama-MacBeth regressions

We provide further supporting evidence using Fama-MacBeth regressions, which allows for adding alternative control variables. The baseline Fama-MacBeth regression model is:

$$RET_{at} = a_t + b_t BETA_{at-1} + c_t BVOL_{at-1} + d_t RET_{at-1} + \eta_{at}, \tag{15}$$

where RET_{at} and RET_{at-1} are the returns of anomaly a at time t and $t-1$, respectively; $BETA_{at-1}$ and $BVOL_{at-1}$ are the market beta and beta volatility of the anomaly a at time $t-1$. To control for the possible impact of the idiosyncratic volatility ($IVOL$) of the anomaly, we add $IVOL_{at-1}$ as a control variable and report the results side-by-side with the basic model in Table 4. Columns (1) and (2) report the regression results for the full sample period. Without the $IVOL$ control, the coefficient of $BVOL$ is 1.008 with a highly significant t-stat of 10.05, and the market beta is insignificant, consistent with the portfolio sort results in Table 2. Prior-month anomaly returns have a significant positive coefficient, which is contrary to the findings for individual stocks (Lehmann, 1990; Jegadeesh, 1990). However, the result here applies to anomaly portfolios of which the composition changes on a monthly basis, and it produces the expected persistence of the anomaly returns.

Adding $IVOL$ as a control reduces both the magnitude and the significance of the coefficient of $BVOL$ (now 0.736 with a t-stat of 4.65) resulting from the positive correlation between $BVOL$ and $IVOL$, with both having a similar impact on future anomaly returns. $IVOL$ also has a significantly positive coefficient. These results are consistent with the fact that both beta uncertainty and idiosyncratic volatility are arbitrage barriers per equation (5). The regression results show that both have distinct effects on future anomaly returns, supporting our argument that beta uncertainty is related to systematic risk while $IVOL$ captures idiosyncratic risk.

We again divide the full sample period into subperiods of high sentiment and low sentiment using the sentiment index of Baker and Wurgler (2006), and run the regressions separately for the two subperiods. Columns (3) and (4) of Table 4 report the results for the high sentiment periods, and columns (5) and (6) reports the results for the low sentiment periods. Without the control for $IVOL$, in columns (3) and (5), the results are consistent with the nonparametric results in Table 3. The $BVOL$ premium in the High Sentiment and Low Sentiment periods are similar to each other and to that in the Full Period. In addition, the persistence of the anomaly returns is similar across the sentiment periods.

Including the control for $IVOL$ in both subsamples in Columns (4) and (6) of Table 4, the regression results are again similar to those for the full sample period except that the effect of $IVOL$ appears stronger in the high sentiment periods while the effect of $BVOL$ appears stronger in the low sentiment periods. An explanation is that the market volatility and expected return, which magnify $BVOL$'s effects but diminish $IVOL$'s effects on anomaly returns,[8] are higher during low sentiment periods (the average monthly volatility and return of the market excess return are 5.46% and 0.81%, respectively, during Low Sentiment periods versus 4.56% and 0.28%, respectively, during High Sentiment periods).

A separate interesting result in Table 4 is that the market beta has a negative and significant coefficient in high sentiment periods, suggesting that positive betting against beta returns (Jensen, Black, and Scholes, 1972; Fama and MacBeth, 1973; Frazzini and Pedersen, 2014) exist in the anomaly portfolios, although these returns are significant only in periods when sentiment is high. This directly challenges Boloorforoosh et al. (2020) who argue that beta uncertainty explains the betting against beta anomaly. This result also departs from Liu, Stambaugh, and Yuan (2018) who find that the betting against beta anomaly becomes insignificant once $IVOL$ is controlled for. The discrepancy may arise because we separate high and low sentiment regimes, whereas Liu et al. (2018) pool these regimes together. In our analysis, the positive beta coefficient during low sentiment (in Column (4)) offsets the

[8]From Equation (5) it follows that $\partial \ln(w^*_{t-1})/\partial\sigma^2_{mt-1} = (\mu^2_{mt-1} + R)/[\sigma^2_{mt-1}(\sigma^2_{mt-1} + \mu^2_{mt-1} + R)]$, and $\partial \ln(w^*_{t-1})/\partial\mu_{mt-1} = -1/\mu_{mt-1} - 2\mu_{mt-1}/(\sigma^2_{mt-1} + \mu^2_{mt-1} + R)$, where $R = \sigma^2_{at-1}/\sigma^2_{\beta at-1}$, both of which depend negatively on $\sigma^2_{\beta at-1}$ ($BVOL$) and positively on σ^2_{at-1} ($IVOL$). So the second partial derivatives, $\partial^2 \ln(w^*_{t-1})/\partial\sigma^2_{\beta at-1}\partial\sigma^2_{mt-1}$ and $\partial^2 \ln(w^*_{t-1})/\partial\sigma^2_{\beta at-1}\partial\mu_{mt-1}$ are negative, while $\partial^2 \ln(w^*_{t-1})/\partial\sigma^2_{at-1}\partial\sigma^2_{mt-1}$ and $\partial^2 \ln(w^*_{t-1})/\partial\sigma^2_{at-1}\partial\mu_{mt-1}$ are positive.

negative beta coefficient during high sentiment (in Column (6)), which is reflected in the insignificant beta coefficient in our pooled results in Column (2).

6 Beta uncertainty and idiosyncratic volatility

The regression results in Table 4 suggest a close relation between $BVOL$ and $IVOL$ and similar effects on anomaly returns, even though we present a detailed model that explicitly separates the two. Given the latent nature of these variables, they are likely inherently correlated, and the estimation procedure may introduce additional correlation. Further, as both are potential arbitrage barriers, their effects on anomaly returns may derive from a similar mechanism. Although the regression results in Table 4 indicate that their effects are distinct, we conduct a more comprehensive analysis in this section to control for $IVOL$, allowing in particular for a possible nonlinear relation between $BVOL$ and $IVOL$.

6.1 Orthogonalization and standardization

Our first analysis in this subsection is to decompose $BVOL$ via regression into components that are related to $IVOL$ and a residual that is orthogonal to $IVOL$ and use the residual in the Fama-MacBeth regressions to test its effect on the anomaly returns. Specifically, we use the following regression to decompose $BVOL$,

$$BVOL_{it} = a_t + b_t IVOL_{it} + c_t IVOL_{it}^2 + \varepsilon_{it}^{BVOL}, \tag{16}$$

We include the squared term of $IVOL$ to capture the possible nonlinear relation between $BVOL$ and $IVOL$. Conversely, we decompose $IVOL$ in the same way to obtain the residual orthogonal to $BVOL$.

We then regress future anomaly returns on the residuals separately, similar to Table 4, and report the results in Panel A of Table 5. Observe that the residual of $BVOL$ (orthogonalized) still has a positive and significant coefficient, indicating a strong positive relation with future anomaly returns, with or without the presence of $IVOL$, albeit the $IVOL$ coefficient is also positive and significant. The opposite holds as well: the orthogonalized residual of $IVOL$ remains significant and positive, while $BVOL$ is also positively and significantly associated with future anomaly returns. These results corroborate the findings in Table 4, and confirm that the effect of $BVOL$ on future anomaly returns is distinct from that of $IVOL$.

To compare numerically the magnitudes of the $BVOL$ and $IVOL$ effects, we cross-sectionally standardize both variables (dividing by their monthly standard deviations across all anomalies after subtracting the cross-sectional means each month) and regress future anomaly returns on the two standardized variables. Panel B of Table 5 reports the regression results, which show again that both variables are positive and significant. However, the coefficient of $BVOL$ (standardized) is considerably larger and more significant than that of the standardized $IVOL$: 0.122 with a t-stat of 4.76 versus 0.084 with a t-stat of 2.85. This implies that the effect of $BVOL$ on future anomaly returns is quantitatively stronger than that of $IVOL$, consistent with the discussion related to equation (6), that, as a result of the diversification inherent in anomaly positions, $BVOL$'s magnitude is substantially (about 50%) larger than $IVOL$'s.

6.2 Double sorting

In this subsection, we employ a double sorting approach to investigate the interactive effect of beta uncertainty and idiosyncratic volatility on the anomaly returns.

We first conduct dependent double sorting. Specifically, we first sort anomalies into five quintiles based on $IVOL$ and then within each $IVOL$ quintile, we sort anomalies again into five quintiles based on $BVOL$, forming 5×5 quintile portfolios. Panel A of Table 6 reports, within each $IVOL$ quintile, the returns of the quintile portfolios and the corresponding spread portfolio between the fifth and first quintiles of $BVOL$. We also report the Fama-French five-factor alphas (FF5 alphas) for the five spread portfolios. In general, the returns of $BVOL$ quintile portfolios increase with the level of $IVOL$, and the spread portfolios follow the same pattern. For example, the FF5 alpha of the $BVOL$ spread portfolio is 0.140% per month in the first $IVOL$ quintile, less than one-fourth that of the spread portfolio in the fifth $IVOL$ quintile, which is 0.589%. Nevertheless, all but the second spread portfolios have positive and significant average returns and FF5 alphas, and even the FF5 alpha of the second spread portfolio is significant at the 10% level. Controlling for the effect of $IVOL$ by taking the averages across the five $IVOL$ quintiles, the spread is 0.268% in return and 0.303% in FF5 alpha, both highly significant. Compared to the single sort results in Table 2, $IVOL$ clearly has an important impact on the predictability of $BVOL$ on future anomaly returns, although the effect of $BVOL$ remains clearly significant after controlling for $IVOL$.

In Panel B of Table 6, we conduct similar double sorting controlling for $BVOL$, instead. We first sort anomalies into five quintiles based on $BVOL$, and then form the 5×5 quintile portfolios of $IVOL$. Of the five spread portfolios of $IVOL$, only two (in quintiles 3 and 4 for $BVOL$) are significant; the other three are insignificant. Overall, controlling for the effect of $BVOL$ by taking the averages across the five $BVOL$ quintiles, the spread is 0.251% in return and 0.182% in FF5 alpha, both significant but smaller in magnitude than the $BVOL$ effect controlled for $IVOL$ in Panel A. Figure 2 illustrates how the mean anomaly returns depend on the order of sorting by $BVOL$ and $IVOL$: the left panel provides the portfolio returns when sorting first by $IVOL$ then by $BVOL$; the right panel when sorting first by $BVOL$ then by $IVOL$. This confirms our previous results that the effects of $BVOL$ and $IVOL$ are intertwined, and that the effect of $IVOL$ is weaker after controlling for the effect of $BVOL$.

Table 7 reports the same double sorting strategies but with three-month and six-month holding periods, respectively. We extend to longer holding periods because the arbitrage-limiting effect of $BVOL$ lasts likely longer than one month, and one month results can be noisy. Indeed, Table 7 shows similar results as in Table 6, but they are more significant. All five spread portfolios of $BVOL$ are highly significant now, although the magnitudes of the returns are largely unchanged. In contrast, again only two of the five spread portfolios of

$IVOL$ are significant.[9]

7 Longevity of anomalies

Our previous analysis provides evidence that anomalies with more beta uncertainty have higher near-future returns, and we argue that beta uncertainty is an arbitrage risk (barrier), preventing correction of mispricing in the anomalies. In this section we examine the anomaly performance over longer horizons.

On the one hand, McLean and Pontiff (2016) document that anomalies decline out of sample, especially after publication. They note that anomalies exhibiting higher in-sample returns attract more capital from arbitrageurs and experience heightened arbitrage activities, leading to faster decay. Building on their findings, we hypothesize that higher past returns result in lower returns over longer horizons due to increased arbitrage activities.

On the other hand, because beta uncertainty functions as an arbitrage barrier, we conjecture that there will be interactions between beta uncertainty and arbitrage activities mediated by past returns: we anticipate that beta uncertainty attenuates the diminishing effect of high past returns on future anomaly returns. This occurs because, when past returns are at a high level where arbitrage activities are prevalent, greater beta uncertainty imposes more significant constraints on arbitrage activities. Conversely, past returns are expected to amplify beta uncertainty. When beta uncertainty is high, arbitrageurs have less incentive to allocate resources if past returns are weak, leading to a weaker marginal impact of beta uncertainty. Consequently, we anticipate that the interaction term of $BVOL$ and past returns has a positive coefficient.

[9] For completeness, we also perform independent double sorting (results available from the authors), which yields largely similar results as in Table 6, with a few differences. First, the returns and FF5 alphas of the $BVOL$ spread portfolios are slightly higher and more significant than for the dependent double sorting. Second, the $IVOL$ spread portfolio in the highest $BVOL$ quintile now has significant but negative returns.

We use the following panel regression to test these hypotheses.

$$
\begin{aligned}
CRET^i_{t+1,t+n} = b_0 + b_1\overline{BETA}^i_{t-m,t-1} + b_2\overline{BVOL}^i_{t-m,t-1} + b_3\overline{IVOL}^i_{t-m,t-1} + b_4 CRET^i_{t-m,t-1} \\
+ b_5(\overline{BVOL} \times CRET)^i_{t-m,t-1} + b_6(\overline{IVOL} \times CRET)^i_{t-m,t-1} + \gamma_i + \lambda_t + \epsilon^i_t
\end{aligned}
\tag{17}
$$

where $CRET^i_{t+1,t+n}$ is the cumulative holding period return over n periods starting at month $t+1$, and $CRET^i_{t-m,t-1}$ is the cumulative measurement period return over m periods ending at month $t-1$. The holding period n ranges from 3 to 36 months; the measuring period for past return m is either 36 or 60 months. $\overline{BETA}^i_{t-m,t-1}$, $\overline{BVOL}^i_{t-m,t-1}$, and $\overline{IVOL}^i_{t-m,t-1}$ are the average market beta, $BVOL$, and $IVOL$ for anomaly i either in the 36- or the 60-month window prior to the formation month t. $(\overline{BVOL} \times CRET)^i_{t-m,t-1}$ and $(\overline{IVOL} \times CRET)^i_{t-m,t-1}$ represent the interaction terms between $\overline{BVOL}^i_{t-m,t-1}$ and $\overline{IVOL}^i_{t-m,t-1}$ with the past cumulative returns $CRET^i_{t-m,t-1}$.

The regression results are reported in Table 8. First, $\overline{BVOL}$ is positive but insignificant in each regression, suggesting that when the past returns are very low, there are no arbitrage activities to begin with, and thus no beta uncertainty effect. Similarly, $\overline{IVOL}$ is insignificant. However, the past return ($CRET$) coefficient is always negative and highly significant, consistent with McLean and Pontiff (2016). This suggests that high past anomaly returns likely reduce future anomaly returns over horizons from 3 months to 36 months.

Nevertheless, beta uncertainty has a significant modulating effect on the diminishing effect of past returns, with a positive coefficient suggesting that the greater the beta uncertainty, the smaller the weakening effect of the past returns on future returns. The significantly positive coefficient on this interaction term suggests that the effect of beta uncertainty is moderated by past returns as well: the higher the past returns, the stronger the arbitrage-reducing impact of beta uncertainty on future anomaly returns. Finally, the interaction term between $IVOL$ and the past returns is always negative but not always significant, implying again that $BVOL$ and $IVOL$ have distinct effects, here on future returns.

For the overall impact of lagged cumulative anomaly returns on future cumulative anomaly returns, we can easily compare the original McLean and Pontiff (2016) component (proxied by the past returns) and the beta uncertainty component in quantitative terms. Focusing on the case of 36 months lagged cumulative returns on 36 months future cumulative returns in Table 8, we have that the McLean and Pontiff (2016) component is -0.090, significantly negative as expected, with t-statistic of -3.03, and the beta uncertainty component is $0.031 \times \overline{BVOL}$, significantly positive as expected, with t-statistic of 5.10. The net effect, if negative, implies anomaly factor mean reversion as in McLean and Pontiff (2016), and, if positive, implies anomaly factor momentum. The size of the effect of the past returns depends on the level of beta uncertainty.

As we measure $\overline{BVOL}$ by the decile rank here, it is straightforward to calculate the quantitative impact. For the lowest $\overline{BVOL}$ deciles, the McLean and Pontiff (2016) component dominates, yielding mean reversion for these anomalies. For the third-lowest decile, we have $\overline{BVOL} = 3$, which generates a net of zero. Anomalies with $\overline{BVOL}$ in deciles higher than the third display momentum. For the anomalies with the largest beta uncertainty, in decile ten, the net effect is 0.220. This is consistent with our earlier result that (only) anomalies with the highest beta uncertainty generate persistently high abnormal returns. For the alternative holding and measurement periods in Table 8, anomaly return momentum occurs for higher beta uncertainty deciles in all cases, starting anywhere from decile two to decile seven.

8 Firm level evidence and investor arbitrage activity

In the preceding analyses, anomalies with high beta uncertainty are found to have high future returns. We argue that beta uncertainty represents an arbitrage risk that prolongs mispricing. Here, we provide further evidence by expanding the testing to the individual firm level, also allowing us to examine the specific investor arbitrage activity. The premise is that, if arbitrage risk from beta uncertainty contributes to the persistence of an anomaly, then one would expect higher beta uncertainty to increase abnormal returns at the micro level. We also expect firm level beta uncertainty to limit investor arbitrage activity, e.g. reducing arbitrage trades when arbitrage risk is high.

8.1 Beta uncertainty enhances firm level mispricing

We perform tests to see if mispricing in individual stocks, rather than portfolios of anomaly positions, with higher beta uncertainty produces stronger future returns due to increased arbitrage risk. This is particularly interesting because of the debate to what extent anomalies reflect mispricing rather than missing risk factors. To this end, we use the mispricing score (MPS) of Stambaugh et al. (2015), which is a score from 1 to 100 constructed from 11 anomalies studied in Stambaugh et al. (2012). We subtract 50 so that stocks with a score higher than 0 are overpriced, and those with a score lower than 0 are underpriced. We argue that, because beta uncertainty creates additional arbitrage risk limiting arbitrage activity, mispricing will be worse (on average) for stocks with higher beta uncertainty. To model this magnifying effect of beta uncertainty on mispricing, we include an interaction term in the following Fama-MacBeth regression:

$$RET_{it+1} = a_t + b_t BETA_{it} + c_t BVOL_{it} + d_t MPS_{it} + g_t (MPS \times BVOL)_{it} + h_t^{'} X_{it} + \epsilon_{it+1}, \quad (18)$$

where RET_{it+1} is the return on stock i in month t+1; MPS is the mispricing score (minus 50) calculated by Stambaugh et al. (2015). $BETA$ and $BVOL$ are the estimated market beta and beta volatility for stock i in month t, respectively; $(MPS \times BVOL)_{it}$ is the interaction term between mispricing and beta uncertainty; X_{it} is a vector of other control variables for stock i in month t which includes LSZ (the log of market equity), LBM (the log book-to-market ratio), RET_{it} (the lagged return), AMD (the stock liquidity measure of Amihud (2002)), and $IVOL$. To match the ranking attribute of MPS and help interpret the coefficients of the interaction terms, the decile ranks of $BVOL$ and $IVOL$ are used in the regression.

Because arbitrage risks are most relevant for overpriced stocks, and our previous results confirm that the effect of beta uncertainty on anomalies is more prominent for the short legs, we run separate regressions for overpriced stocks ($MPS > 0$) and underpriced stocks ($MPS < 0$) and report the results in Table 9. Because of the requirements of our estimation, demanding a relatively long history (50 years), stocks in our sample are mostly mature and potentially large.

In the first regressions of Panels A and B, we include $BVOL$ and MPS, ignoring the cross product. The results confirm those of Stambaugh et al. (2015) as well as those of Armstrong et al. (2013) and Hollstein et al. (2020). For both the overpriced and underpriced stocks, the MPS impact is negative and statistically significant, -0.023 with a t-stat of -5.36 in the overpriced stocks, versus -0.017 with a t-stat of -3.08 in the underpriced stocks. Moreover, the $BVOL$ coefficient is significantly negative, reproducing the results of Armstrong et al. (2013) and Hollstein et al. (2020) no matter if stocks are overpriced or underpriced, with coefficient -0.029 and t-stat of -2.07 for the overpriced stocks and coefficient -0.016 with t-stat of -1.92 for the underpriced stocks.

We then add the interaction term ($MPS \times BVOL$) to the second regressions in Panels A and B and find different results between the overpriced and underpriced stocks. For both the overpriced and underpriced stocks, $BVOL$ is insignificant, different now from Armstrong et al. (2013). Crucially, for the overpriced stocks, the interaction term is negative and significant, consistent with the hypothesis that beta uncertainty is an arbitrage barrier exacerbating the mispricing in overpriced stocks. In contrast, both $BVOL$ and its interaction with MPS are insignificant for the underpriced stocks, as anticipated: amplified arbitrage constraints are less relevant when correction of mispricing does not require shortselling.

Lastly, we include $IVOL$ and ($MPS \times IVOL$) to the third regressions in Panels A and B, as further arbitrage cost measures. While the signs are negative, the coefficients are insignificant, suggesting again that the beta uncertainty is a quantitatively more important indicator of arbitrage impediments. In all specifications in Table 9 both size (LSZ) and book-to-market (LBM) have significant coefficients with the expected signs, indicating that our sample, although limited, is representative. Further, $BETA$ becomes positive and significant for the underpriced stocks, but insignificant or marginally significant for the overpriced stocks, similar in sign to the findings of Liu et al. (2018) who double sort stocks by MPS and beta and find beta is positively (but insignificantly) related to future returns for underpriced stocks.

These results provide empirical evidence against the explanation of the impact of beta uncertainty in Armstrong et al. (2013) who offer a partial equilibrium model in which a firm's expected return decreases in the factor-loading uncertainty because the covariance between a firm's cash flows and the pricing kernel, and hence the price of these cash flows, is a convex function of the firm's future risk factor loading. In contrast, we argue that beta uncertainty works as an arbitrage risk and interacts with mispricing to affect future returns only when shorting the stock is required for arbitrage. Without properly controlling for the interaction term of beta uncertainty and mispricing as well as the mispricing sign, it appears that beta uncertainty is negatively related to future returns. However, once we correctly include the interaction term, the effect of beta uncertainty is no longer negative.

8.2 Beta uncertainty and arbitrage activities

To validate our inference that the impact of beta uncertainty on future anomaly returns indeed stems from impeding arbitrage, we seek direct evidence that beta uncertainty reduces investor arbitrage activity. To capture investor arbitrage transactions, we follow Hanson and Sunderam (2014) and use firm level monthly short interest as a proxy for arbitrageur positions. Monthly short-selling interest of each firm is defined as the number of shares shorted divided by the total number of shares outstanding for every month. We focus on the short-side activity as arbitrage-related for the following reasons: first, the short-selling data is widely available; second, short-selling involves more constraints and risks (lacking even a limited liability bound) and is more likely to be initiated by sophisticated investors (arbitrageurs) who are actively looking for alpha; third, most of the anomaly research shows that anomaly strategy profits are often generated from the short legs. We use the following Fama-MacBeth regression model to test our hypothesis:

$$SI_{it+1} = a_t + b_t BVOL_{it} + c_t IVOL_{it} + d_t^{'} X_{it} + \epsilon_{it+1}, \tag{19}$$

where the dependent variable SI_{it+1} is the short-selling interest in month $t+1$ for stock i. X_{it} is a vector of firm characteristics for stock i in month t as controls, including LSZ, LBM, past six month of returns $RET_{-6,-1}$, and the percentage of institutional ownership IOR. If beta uncertainty indeed discourages arbitrage, we expect the coefficient of $BVOL_{it}$ to be negative.

Panel A of Table 10 reports the regression results for next period short-selling interest, SI_{it+1}; Panel B reports the results for the contemporaneous short-selling interest, SI_{it}. As the two sets of results are very similar, we focus on the results for SI_{it+1} in Panel A. Column (1) only includes $BVOL$ and $IVOL$; Columns (2) and (3) add more control variables. In all three models, $BVOL_{it}$ is always negative and statistically significant at the 1% level. For example, the coefficient in Column (3), which includes all controls, is -0.049 (t-stat$=-4.96$). Virtually, the same results hold in Panel B. The evidence supports our main argument that beta uncertainty creates arbitrage barriers and dampens arbitrageur short-selling activities, both for the same period and for the subsequent period.

In sharp contrast to $BVOL$, however, $IVOL$ always has a positive and significant coefficient. For example, the coefficient of $IVOL$ in Column (3) is 0.197 (t-stat=8.74). The significantly positive coefficient implies that $IVOL$ is associated with increases in short-selling activities. This puzzling finding is in line with the idiosyncratic volatility puzzle (Ang et al., 2006, 2009) in which stocks with high idiosyncratic volatility yield low returns; smart investors short sell these stocks to profit from this anomaly. $IVOL$ also proxies for the degree of divergence in investor opinions so that, based on Miller (1977), the high $IVOL$ stocks become more overpriced, attracting short selling by arbitrageurs.

Overall, Table 10 provides incremental evidence supporting the hypothesis that high beta uncertainty increases the barrier to arbitrage, leading to a low level of arbitrage transactions.

8.3 An alternative measure for arbitrage activity

We evaluate the sensitivity of the results to short interest as the proxy for arbitrage activity by examining an alternative measure of arbitrage activity proposed by Lou and Polk (2022): co-momentum. Higher co-momentum means that abnormal returns (returns adjusted for industry and Fama-French three-factor risk) of the winner decile (potentially underpriced) and the loser decile (potentially overpriced) become more correlated on average. This signifies more arbitrage activity as increased investment, in the same direction by decile, occurs with the intent to exploit mispricing, leading to related return movements in these deciles. Table 11 shows how $BVOL_t$ and $IVOL_t$, averaged over the winner and loser deciles, affect the arbitrage activity, proxied by co-momentum, over time. The results are similar for the current co-momentum $CoMOM_t$ in Panel A and the next-month co-momentum $CoMOM_{t+1}$ in Panel B: The average $BVOL$ significantly lowers arbitrage activity and the average $IVOL$ significantly raises arbitrage activity. These impacts are fully consistent with Table 10 in which short interest instead of comomentum is used as the proxy of arbitrage activity. When we add controls for the averages of beta, firm size, and turnover, the $BVOL$ effect on arbitrage activity remains significant but the $IVOL$ impact becomes insignficicant for both current and next-month co-momentum.

8.4 Comparison with alternative explanations

Sections 7 and 8 of the paper provide supplementary information on the impact of $BVOL$ separate from its direct effect on monthly anomaly returns. The supplementary findings are in all cases consistent with the level of $BVOL$ representing a degree of arbitrage impediment.

An alternative explanation for the importance of $BVOL$ is that it represents factor risk. Boloorforoosh et al. (2020), in particular, views $BVOL$ as a Merton risk factor. In the Fama-MacBeth results of section 5.3 $BVOL$ may be viewed as loadings on a factor mimicking portfolio of the test assets (the set of anomalies). Regarding $BVOL$ either as an asset's attribute or as its risk factor loading is observationally equivalent with respect to return impact.[10] To distinguish a generic systematic risk explanation from our restricted arbitrage view, only the supplementary results are pertinent.

We find that the impact of cumulative previous returns on anomaly returns in Table 8 is mitigated by $BVOL$. Explaining this by factor risk would require an elaborate theory of changes in loadings over time. Similarly, the impact of $BVOL$ in Table 9, as mediated by asset mispricing, is difficult to explain by loading changes on an aggregate $BVOL$-mimicking risk factor. Lastly, under a risk explanation, the return effects from $BVOL$ exposure are compensation for risk and would not incite arbitrage trading. If arbitrage is not involved, there are no easy explanations for why the factor loadings relate to either short-selling interest or co-momentum, as any such explanation is necessary to explain the results in Tables 10 and 11.

The finding that $BVOL$ directly lowers arbitrage activity, as proxied either by short interest or co-momentum, validates the theory that $BVOL$ raises anomaly returns specifically through the arbitrage barrier channel. Anomaly returns increase because higher anomaly $BVOL$ reduces arbitrage activity, leaving higher anomaly returns. Other theories of a $BVOL$ impact fail to explain the reduced arbitrage activity and/or are contradicted by the findings: Miller (1977) may be interpreted as implying a role for $BVOL$ as one of the causes of investor disagreement. Higher disagreement causes more overpricing, affecting prospective returns, but this would imply more arbitrage activity by shortsellers, not less. Armstrong et al. (2013) argues that $BVOL$ implies a positive Jensen's inequality effect because stock prices

[10] Fama (1976) points out that the OLS coefficient in the Fama-MacBeth regression equates to the return on a weighted average portfolio of zero-investment test assets. Balvers and Luo (2018) emphasize the implication that characteristics and factor loadings cannot be distinguished based on return data because for every characteristic a factor-mimicking portfolio may be constructed that generates the characteristic of each asset as its loadings.

are convex in the stock's beta. Higer $BVOL$ then implies lower prospective returns, instead of higher, and, in addition, has no impact on arbitrage activity.

9 Conclusion

We propose that uncertainty surrounding risk loadings – beta uncertainty – serves as a significant barrier to arbitrage. In a simple market model with uncertainty about market betas, the demand by investors wishing to take advantage of positions with positive alpha is dampened because the investors are not able to transport these alphas without incurring an unpredictable degree of systematic risk. Fully hedging this risk would require offsetting the uncertain factor loading of the target asset by shorting a portfolio, also with uncertain factor loading. This difficulty in hedging the systematic risk accompanying positive alpha positions hinders active investment, thereby reducing arbitrage of the target asset.

In anomaly positions, typically involving a portfolio of many different stocks, arbitrage risk is diminished due to reduced idiosyncratic risk, which by definition is easily diversifiable. However, the beta uncertainty becomes relatively more critical because beta fluctuations may be positively correlated. Arbitrage then becomes precarious because random risk loading may provide poor returns at a bad time. For portfolios targeting an anomaly, with evident active investment opportunities, we claim that beta uncertainty enhances anomaly returns and reduces arbitrage, and that the quantitative impact is stronger than that of a chief alternative arbitrage barrier, idiosyncratic volatility.

To test the hypotheses involving the impact of beta uncertainty on arbitrage incentives we focus first on anomaly positions. Empirically, we utilize the 207 long-short anomaly portfolios compiled by Chen and Zimmermann (2022). Because beta uncertainty and idiosyncratic risk are correlated and function similarly as arbitrage risks, we propose a Bayesian market model to explicitly capture beta uncertainty and the idiosyncratic risk of anomaly portfolio returns, In the model, anomaly beta is stochastic, following an AR(1) process, and idiosyncratic volatility is stochastic, following a log AR(1) process. Accordingly, two separate random processes drive the beta uncertainty and idiosyncratic risk.

We find that our measure of beta uncertainty, $BVOL$, impacts anomaly returns substantially. The anomaly decile with the highest $BVOL$ generates an annualized return of 8.7 percentage points higher than the decile with the lowest $BVOL$, even after adjustment with the five Fama-French risk factors. Only the quintile of anomalies with the highest $BVOL$ generates significant returns. The impact of $BVOL$ on anomaly returns is stronger and more robust than that of $IVOL$.

We know from McLean and Pontiff (2016) that anomaly returns are persistent but decay more at higher return levels. If $BVOL$ operates as an arbitrage hurdle, a higher uncertainty level should slow the decay. We expect the $BVOL$ effect on future anomaly returns to positively depend to the current size of the anomaly returns. The results indeed shows that the interaction between $BVOL$ and cumulative anomaly returns over the past 3 years (or 5 years) has a significantly positive coefficient for holding periods from 3 months to 3 years. By itself, past cumulative returns have a negative impact on future returns, suggesting mean reversion in anomaly returns. But for anomalies with sufficiently high $BVOL$, the mean reversion changes to momentum.

To compare against previous literature linking beta uncertainty (as a separate risk factor or a measure of convexity impact) to asset returns and to obtain a micro perspective on the arbitrage impediments, we consider also the importance of $BVOL$ for arbitrage of individual stocks. We use the mispricing measure for each stock, as defined by Stambaugh et al. (2015), MPS, and find that, for overpriced stocks, the interaction term $BVOL \times MPS$ is significantly negative: the greater the overpricing, the more incentive for arbitrage and the more critical the cost imposed by beta uncertainty. Without the interaction term, our results replicate the findings of Armstrong et al. (2013): the $BVOL$ coefficient is significantly negative (for both overpriced and underpriced samples). However, their explanation that the stock's expected return depends negatively on beta uncertainty because the stock price is a convex function of its beta, does not depend on mispricing. The significance of the $BVOL$ coefficient should not disappear when we add the interaction terms. The fact that it does, favors our arbitrage-barrier explanation.

To produce separate, and more direct, evidence validating $BVOL$ as an arbitrage barrier, we employ proxies for arbitrage activity and examine if they are indeed affected by $BVOL$. We utilize short interest and comomentum as indicators for the level of arbitrage in a stock, obtaining similar results in both cases: higher $BVOL$ significantly and negatively impacts short interest as well as comomentum, thus pinpointing restricted arbitrage by both of these measures. These findings provide a tangible indication that the impact of $BVOL$ on anomaly returns works indeed through the channel of arbitrage deterrence.

In summary, beta uncertainty is theoretically a significant hurdle for arbitrage and expected to have a quantitatively more important impact than idiosyncratic volatility in determining anomaly returns. Given a measure of beta uncertainty that accounts for time variation as well as estimation risk in market betas, we find that indeed anomaly returns vary economically and statistically significantly with beta uncertainty and more so than with the idiosyncratic risk of the anomalies. At the individual stock level we obtain corresponding results for the impact of beta uncertainty on mispricing and returns, and also provide evidence that beta uncertainty directly reduces arbitrage activity.

References

Amihud, Yakov, 2002, Illiquidity and stock returns: Cross-section and time-series effects, *Journal of Financial Markets* 5, 31–56.

Ang, Andrea, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang, 2006, The cross-section of volatility and expected returns, *Journal of Finance* 61, 259–299.

Ang, Andrew, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang, 2009, High idiosyncratic volatility and low returns: International and further U.S. evidence, *Journal of Financial Economics* 91, 1–23.

Armstrong, Christopher S., Snehal Banerjee, and Carlos Corona, 2013, Factor-loading uncertainty and expected returns, *Review of Financial Studies* 26, 158–207.

Baker, Malcolm, and Jeffrey Wurgler, 2006, Investor sentiment and the cross-section of stock returns, *Journal of Finance* 61, 1645–1680.

Baker, Malcolm, Jeffrey Wurgler, and Yu Yuan, 2012, Global, local, and contagious investor sentiment, *Journal of Financial Economics* 104, 272–287, Special Issue on Investor Sentiment.

Balvers, Ronald J, and H Arthur Luo, 2018, Distinguishing factors and characteristics with characteristic-mimicking portfolios, *Available at SSRN 3203031* .

Barahona, Ricardo, Joost Driessen, and Rik Frehen, 2021, Can unpredictable risk exposure be priced?, *Journal of Financial Economics* 139, 522–544.

Barroso, Pedro, and Andrew Detzel, 2021, Do limits to arbitrage explain the benefits of volatility-managed portfolios?, *Journal of Financial Economics* 140, 744–767.

Bidarkota, Prasad V., Brice V. Dupoyet, and J. Huston McCulloch, 2009, Asset pricing with incomplete information and fat tails, *Journal of Economic Dynamics and Control* 33, 1314–1331.

Boloorforoosh, Ali, Peter Christoffersen, Mathieu Fournier, and Christian Gouriéroux, 2020, Beta risk in the cross-section of equities, *Review of Financial Studies* 33, 4318–4366.

Cao, Jie, and Bing Han, 2016, Idiosyncratic risk, costly arbitrage, and the cross-section of stock returns, *Journal of Banking and Finance* 73, 1–15.

Chen, Andrew Y., 2021, The limits of p-hacking: Some thought experiments, *Journal of Finance* 76, 2447–2480.

Chen, Andrew Y., and Tom Zimmermann, 2022, Open source cross-sectional asset pricing, *Critical Finance Review* 27, 207–264.

Chen, Yong, Bing Han, and Jing Pan, 2021, Sentiment trading and hedge fund returns, *Journal of Finance* 76, 2001–2033.

Chib, Siddhartha, Federico Nardari, and Neil Shephard, 2002, Markov chain Monte Carlo methods for generalized stochastic volatility models, *Journal of Econometrics* 108, 281–316.

Chib, Siddhartha, Federico Nardari, and Neil Shephard, 2006, Analysis of high dimensional multivariate stochastic volatility models, *Journal of Econometrics* 134, 271–341.

Chu, Yongqiang, David Hirshleifer, and Liang Ma, 2020, The causal effect of limits to arbitrage on asset pricing anomalies, *The Journal of Finance* 75, 2631–2672.

Da, Rui, Stefan Nagel, and Dacheng Xiu, 2023, The statistical limit of arbitrage, *University of Chicago Working Paper* .

DeLong, J Bradford, Andrei Shleifer, Lawrence H Summers, and Robert J Waldmann, 1990, Noise trader risk in financial markets, *Journal of Political Economy* 98, 703–738.

DeMiguel, Victor, Alberto Martín-Utrera, and Francisco J Nogales, 2015, Parameter uncertainty in multiperiod portfolio optimization with transaction costs, *Journal of Financial and Quantitative Analysis* 50, 1443–1471.

Duan, Ying, Gang Hu, and R David McLean, 2010, Costly arbitrage and idiosyncratic risk: Evidence from short sellers, *Journal of Financial Intermediation* 19, 564–579.

Engelberg, Joseph, R. David McLean, and Jeffrey Pontiff, 2018, Anomalies and news, *Journal of Finance* 73, 1971–2001.

Fama, Eugene F., 1976, *Foundations of Finance* (Basic Books, New York).

Fama, Eugene F., and James D. MacBeth, 1973, Risk, return, and equilibrium: Empirical tests, *Journal of Political Economy* 81, 607–636.

Frazzini, Andrea, and Lasse Heje Pedersen, 2014, Betting against beta, *Journal of Financial Economics* 111, 1–25.

Garlappi, Lorenzo, Raman Uppal, and Tan Wang, 2007, Portfolio selection with parameter and model uncertainty: A multi-prior approach, *Review of Financial Studies* 20, 41–81.

Gromb, Denis, and Dimitri Vayanos, 2010, Limits of arbitrage, *Annual Review of Finance and Economics* 2, 251–275.

Han, Yufeng, 2006, Asset allocation with a high dimensional latent factor stochastic volatility model, *Review of Financial Studies* 19, 237–271.

Han, Yufeng, Ting Hu, and David A. Lesmond, 2015, Liquidity biases and the pricing of cross-sectional idiosyncratic volatility around the world, *Journal of Financial and Quantitative Analysis* 50, 1269–1292.

Han, Yufeng, and David Lesmond, 2011, Liquidity biases and the pricing of cross-sectional idiosyncratic volatility, *Review of Financial Studies* 24, 1590–1629.

Han, Yufeng, Yueliang Lu, Weike Xu, and Guofu Zhou, 2024, Mispricing and anomalies: An exogenous shock to short selling from JGTRRA, *Journal of Empirical Finance* forthcoming.

Hanson, Samuel G., and Adi Sunderam, 2014, The Growth and Limits of Arbitrage: Evidence from Short Interest, *Review of Financial Studies* 27, 1238–1286.

Hollstein, Fabian, Marcel Prokopczuk, and Chardin Wese Simen, 2020, Beta uncertainty, *Journal of Banking and Finance* 116, 105834.

Jegadeesh, Narasimhan, 1990, Evidence of predictable behavior of security returns, *Journal of Finance* 45, 881–898.

Jegadeesh, Narasimhan, and Sheridan Titman, 1993, Returns to buying winners and selling losers: Implications for stock market efficiency, *Journal of Finance* 48, 65–91.

Jensen, Michael C, Fischer Black, and Myron S Scholes, 1972, The capital asset pricing model: Some empirical tests, in Michael C Jensen, ed., *Studies in the theory of capital markets*, 79–121 (Praeger Publishers Inc, New York).

Jones, Charles M, and Owen A Lamont, 2002, Short-sale constraints and stock returns, *Journal of Financial Economics* 66, 207–239.

Kan, Raymond, and Guofu Zhou, 2007, Optimal portfolio choice with parameter uncertainty, *Journal of Financial and Quantitative Analysis* 42, 621–656.

Kim, Sangjoon, Neil Shephard, and Siddhartha Chib, 1998, Stochastic volatility: Optimal likelihood inference and comparison with ARCH models, *Review of Economic Studies* 65, 361–393.

Lam, FY Eric C, and KC John Wei, 2011, Limits-to-arbitrage, investment frictions, and the asset growth anomaly, *Journal of Financial Economics* 102, 127–149.

Lassance, Nathan, and Alberto Martin-Utrera, 2024, Do limits to arbitrage explain portfolio gains from asset mispricing?, *Available at SSRN* .

Lassance, Nathan, Alberto Martín-Utrera, and Majeed Simaan, 2024, The risk of expected utility under parameter uncertainty, *Management Science* forthcoming.

Lehmann, Bruce N, 1990, Fads, martingales, and market efficiency, *Quarterly Journal of Economics* 105, 1–28.

Lewellen, Jonathan, and Jay Shanken, 2002, Learning, asset-pricing tests, and market efficiency, *Journal of Finance* 57, 1113–1145.

Liu, Jianan, Robert F. Stambaugh, and Yu Yuan, 2018, Absolving beta of volatility's effects, *Journal of Financial Economics* 128, 1–15.

Lochistoer, Lars A., and Paul C. Tetlock, 2020, What drives anomaly returns?, *Journal of Finance* 75, 1417–1455.

Lou, Dong, and Christopher Polk, 2022, Comomentum: Inferring arbitrage activity from return correlations, *Review of Financial Studies* 35, 3272–3302.

McLean, R David, 2010, Idiosyncratic risk, long-term reversal, and momentum, *Journal of Financial and Quantitative Analysis* 45, 883–906.

McLean, R. David, and Jeffrey Pontiff, 2016, Does academic research destroy stock return predictability?, *Journal of Finance* 71, 5–32.

Miller, Edward M, 1977, Risk, uncertainty, and divergence of opinion, *Journal of Finance* 32, 1151–1168.

Muravyev, Dmitriy, Neil D Pearson, and Joshua Matthew Pollet, 2022, Anomalies and their short-sale costs, *Available at SSRN 4266059* .

Nagel, Stefan, 2005, Short sales, institutional investors and the cross-section of stock returns, *Journal of Financial Economics* 78, 277–309.

Newey, Whitney K., and Kenneth D. West, 1987, A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–708.

Omori, Yasuhiro, Siddhartha Chib, Neil Shephard, and Jouchi Nakajima, 2007, Stochastic volatility with leverage: Fast and efficient likelihood inference, *Journal of Econometrics* 140, 425–449.

Pitt, Michael K, and Neil Shephard, 1999, Filtering via simulation: Auxiliary particle filter, *Journal of the American Statistical Association* 94, 590–599.

Pontiff, Jeffrey, 2006, Costly arbitrage and the myth of idiosyncratic risk, *Journal of Accounting and Economics* 42, 35–52.

Shleifer, Andrei, and Robert W Vishny, 1997, The limits of arbitrage, *Journal of Finance* 52, 35–55.

Stambaugh, Robert F, Jianfeng Yu, and Yu Yuan, 2012, The short of it: Investor sentiment and anomalies, *Journal of Financial Economics* 104, 288–302.

Stambaugh, Robert F., Jianfeng Yu, and Yu Yuan, 2015, Arbitrage asymmetry and the idiosyncratic volatility puzzle, *Journal of Finance* 70, 1903–1948.

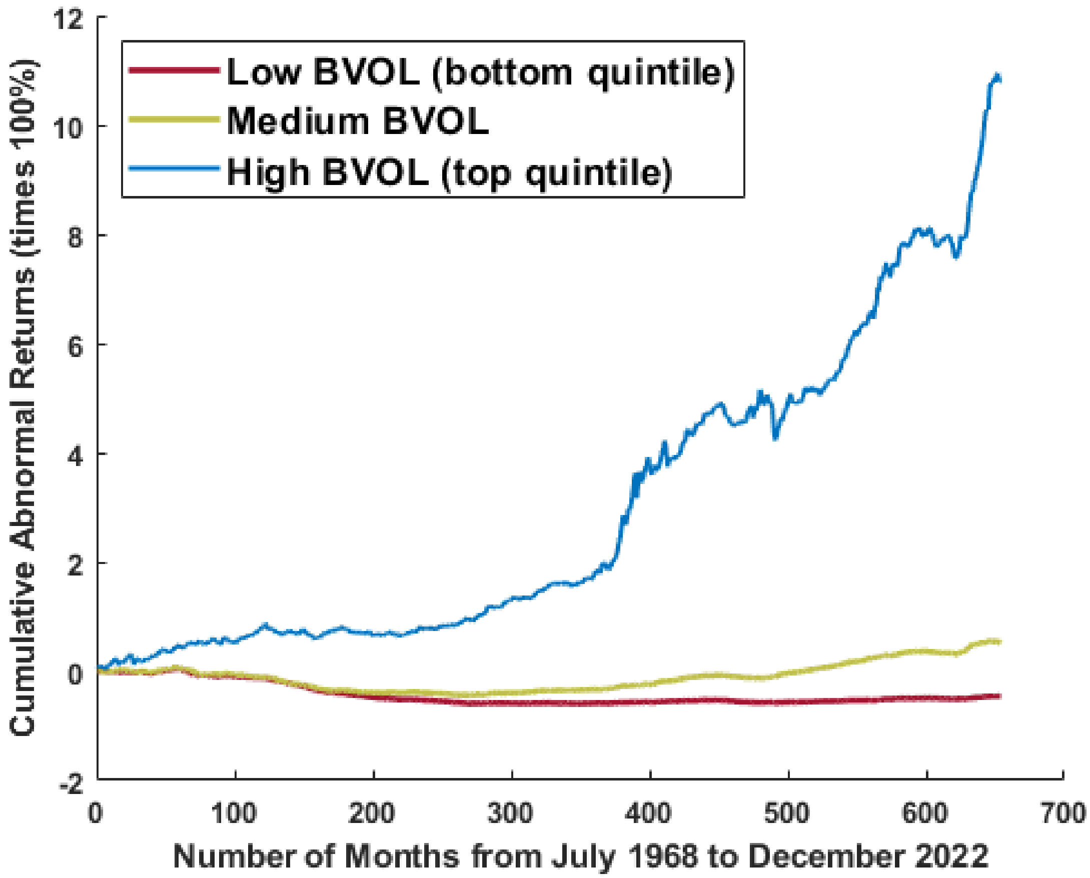

Figure 1: Cumulative abnormal returns from risk-adjusted anomalies sorted by BVOL

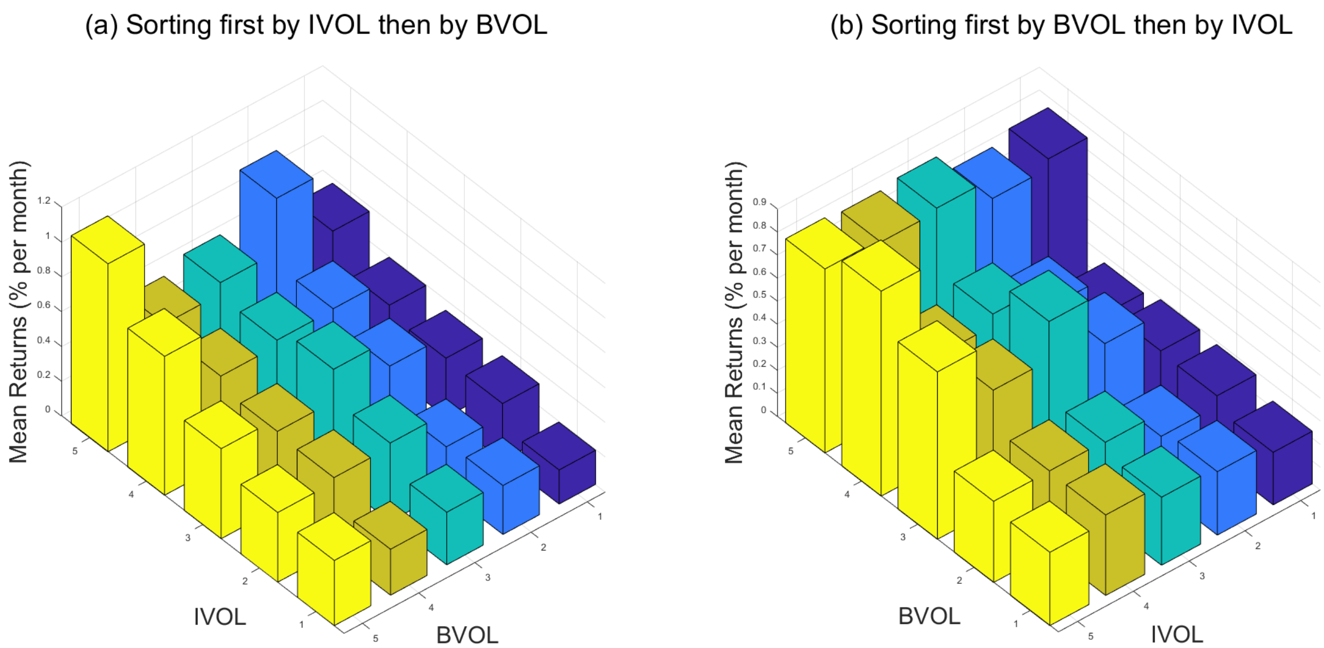

Figure 2: Mean returns of anomaly portfolios sorted by BVOL and IVOL

Table 1: Summary Statistics of Anomaly Portfolios

In this table, we report the summary statistics of the anomaly monthly returns (RET) in percentage terms, the market β ($BETA$), the volatility of β ($BVOL$), and the idiosyncratic volatility of returns ($IVOL$) in Panel A. We first compute the time-series averages of the variables for each anomaly, and then compute the summary statistics across all the anomalies. Panel B presents the distribution of the t-statistics of the market β across the anomalies. A total of 207 anomalies are included and the sample period is from January 1946 to December 2022.

Panel A: Anomaly Market Betas

t-value	Count	Average $BETA$
$t > 2$	56	0.145
$\lvert t\rvert < 2$	7	-0.002
$t < -2$	144	-0.157

Panel B: Summary Statistics

	Mean	Std Dev	Median	Min	Max
$RET(\%)$	0.453	0.465	0.375	-0.488	4.120
$BETA$	-0.070	0.219	-0.052	-0.735	1.211
$BVOL$	0.220	0.153	0.186	0.020	0.982
$IVOL$	0.027	0.015	0.024	0.007	0.123

Table 2: Beta Volatility and Market Beta Sorted Decile Portfolios

At the end of each month, we sort 207 anomaly portfolios to form equal-weighted decile portfolios based on ranking by beta volatility ($BVOL$) and market beta ($BETA$), respectively. We report the time-series average return for each decile portfolio and the high-low $(10-1)$ portfolio. We also report the Fama-French five-factor model (FF5) risk-adjusted returns. Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is denoted by ***, **, and *, respectively. The sample period is from February 1948 to December 2022.

Rank	$BVOL$	$BETA$
1	0.265	0.427
2	0.339	0.544
3	0.255	0.409
4	0.383	0.525
5	0.484	0.531
6	0.596	0.516
7	0.630	0.446
8	0.455	0.436
9	0.618	0.505
10	0.954	0.636
$10-1$	0.689	0.209
	(6.72)***	(0.89)
FF5	0.695	0.063
	(4.96)***	(0.41)

Table 3: Long versus Short Legs and High versus Low Sentiment

At the end of each month, we separately present the long and short leg returns of the 207 anomaly portfolios sorted into equal-weighted decile portfolios by the rankings of beta volatility ($BVOL$). We report the time-series average return for each decile portfolio and the high-low ($10-1$) portfolio return in the full sample period and in periods of high (low) sentiment, which is from Baker and Wurgler (2006) (BW). A month is categorized as high (low) sentiment if the sentiment index is above (below) the median value. Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is indicated by ***, **, and *, respectively. The sample period is from February 1948 to December 2022 for the full sample period in Panel A; the BW index is available from July 1965 to June 2022 in Panels B and C.

	Panel A: Full Period			**Panel B: High Sentiment**			**Panel C: Low Sentiment**		
Rank	Long-Short	Short	Long	Long-Short	Short	Long	Long-Short	Short	Long
1	0.265	0.970	1.235	0.404	0.521	0.925	0.179	1.246	1.425
2	0.339	0.984	1.324	0.402	0.629	1.031	0.301	1.202	1.503
3	0.255	0.976	1.231	0.379	0.609	0.987	0.180	1.200	1.380
4	0.383	0.912	1.295	0.456	0.538	0.994	0.338	1.141	1.479
5	0.484	0.850	1.334	0.577	0.498	1.074	0.428	1.065	1.493
6	0.596	0.746	1.341	0.828	0.352	1.180	0.459	0.978	1.437
7	0.630	0.716	1.346	0.955	0.202	1.157	0.431	1.030	1.461
8	0.455	0.853	1.308	0.672	0.406	1.079	0.322	1.127	1.448
9	0.618	0.712	1.330	0.629	0.351	0.980	0.611	0.932	1.543
10	0.954	0.505	1.459	1.160	-0.092	1.068	0.827	0.872	1.699
$10-1$	0.689	-0.465	0.224	0.756	-0.613	0.142	0.647	-0.374	0.274
	(6.72)***	(-7.45)***	(3.72)***	(4.18)***	(-5.32)***	(1.40)	(5.27)***	(-5.23)***	(3.68)***
FF5	0.695	-0.471	0.224	0.705	-0.511	0.194	0.680	-0.433	0.247
	(4.96)***	(-5.86)***	(2.91)***	(3.09)***	(-3.84)***	(1.68)*	(4.09)***	(-4.89)***	(2.41)**

Table 4: Fama-MacBeth Regression: Beta Volatility and Idiosyncratic Volatility

We report the Fama-MacBeth regression results based on the following basic model:

$$RET_{at} = a_t + b_t BETA_{at-1} + c_t BVOL_{at-1} + d_t RET_{at-1} + \eta_{at},$$

where RET_{at} and RET_{at-1} are the returns of the anomaly long-short portfolio a at time t and $t-1$, respectively; $BETA_{at-1}$ and $BVOL_{at-1}$ are the market beta and beta volatility of the anomaly a at time $t-1$. To control for the impact of the idiosyncratic volatility ($IVOL_{at-1}$) of the anomaly, we also include $IVOL$ as an additional control variable and report the results side-by-side with the basic model. The sentiment index is from Baker and Wurgler (2006), and a median value is used to separate the whole sample period into the high/low sentiment sub-sample periods. Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is indicated by ***, **, and *, respectively. There are a total of 207 anomalies included in the regression. The sample period is from February 1948 to December 2022 for the full sample period; the BW index is available from July 1965 to June 2022.

	Full Period		High Sentiment		Low Sentiment	
	(1)	(2)	(3)	(4)	(5)	(6)
$BETA$	-0.110	-0.119	-0.756**	-0.768**	0.514*	0.492
	(-0.61)	(-0.66)	(-2.45)	(-2.48)	(1.71)	(1.64)
$BVOL$	1.008***	0.736***	0.910***	0.483*	0.950***	0.609**
	(10.05)	(4.65)	(5.55)	(1.80)	(5.60)	(2.48)
$IVOL$		4.703***		7.076**		5.800**
		(2.71)		(2.17)		(2.48)
RET_{-1}	0.115***	0.118***	0.138***	0.138***	0.128***	0.131***
	(8.60)	(8.98)	(6.24)	(6.22)	(5.93)	(6.31)
Intercept	0.115***	0.0412	0.217***	0.109*	0.126***	0.0378
	(3.78)	(1.24)	(4.57)	(1.87)	(2.70)	(0.76)
Adj. R^2	0.288	0.306	0.276	0.295	0.279	0.299

Table 5: Beta Volatility, Idiosyncratic Volatility, and Anomaly Strength: Orthogonalization and Standardization

In Panel A, we first run the following cross-sectional regression each month:

$$BVOL_{at} = a_t + b_t IVOL_{at} + c_t IVOL_{at}^2 + \varepsilon_{at}^{BVOL},$$

and the residuals ε_{at}^{BVOL} are labeled as the orthogonalized $BVOL$ ($Orth.\ BVOL$). We then use $Orth.\ BVOL$ in the Fama-MacBeth regression for the anomaly long-short portfolio returns similar to the ones in Table 4. We use the same approach to orthogonalize $IVOL$ against $BVOL$. The results of the regressions are reported in Panel A. In Panel B, we first standardize the estimated $BVOL$ and $IVOL$, dividing by their monthly standard deviation across all anomalies, then run the Fama-MacBeth regression. Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is indicated by ***, **, and *, respectively. A total of 207 anomalies is included in the regression. The sample period is from February 1948 to December 2022.

Panel A: Orthogonalized $BVOL$ and $IVOL$

$Intercept$	RET_{-1}	$BETA$	$Orth.\ BVOL$	$IVOL$
0.399	0.142	-0.149	0.723	
(15.14)***	(8.95)***	(-0.66)	(3.95)***	
0.103	0.138	-0.186	0.726	10.929
(2.96)***	(8.85)***	(-0.81)	(3.96)***	(9.84)***

$Intercept$	RET_{-1}	$BETA$	$BVOL$	$Orth.IVOL$
0.393	0.145	-0.196		6.458
(14.88)***	(9.19)***	(-0.85)		(3.96)***
0.175	0.137	-0.196	0.931	6.356
(5.23)***	(8.80)***	(-0.85)	(7.92)***	(3.87)***

Panel B: Standardized $BVOL$ and $IVOL$

$Intercept$	RET_{-1}	$Std\ BETA$	$Std.\ BVOL$	$Std.\ IVOL$
0.413	0.114	-0.023	0.122	0.084
(15.61)***	(8.52)***	(-0.43)	(4.76)***	(2.85)***

Table 6: Double Sorting by Beta Uncertainty and Idiosyncratic Volatility

At the end of each month, we first sort anomaly portfolios into quintiles based on idiosyncratic volatility (beta volatility) and then within each quintile, we further sort the anomalies into quintiles based on beta volatility (idiosyncratic volatility), which results in $5 \times 5 = 25$ portfolios. We report the equal-weighted average returns and the Fama-French five-factor model (FF5) risk-adjusted returns for all quintile portfolios and the high-low portfolios. Panel A presents the results of first sorting on idiosyncratic volatility and then on beta volatility; Panel B presents the results of first sorting on beta volatility and then on idiosyncratic volatility. The final row in each panel reports the averages across the five quintiles sorted by $IVOL$ (Panel A) or $BVOL$ (Panel B). Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is indicated by an ***, and **, and an *, respectively. A total of 207 anomalies is included in the sample. The sample period is from February 1948 to December 2022.

Panel A: First on $IVOL$ then on $BVOL$

		$BVOL$								
		1	2	3	4	5	High-Low		FF5	
$IVOL$	1	0.196	0.281	0.302	0.264	0.374	0.178	(4.41)***	0.140	(3.44)***
	2	0.328	0.254	0.450	0.426	0.400	0.072	(1.17)	0.125	(1.69)*
	3	0.339	0.468	0.624	0.442	0.517	0.178	(2.20)**	0.215	(2.27)**
	4	0.394	0.548	0.542	0.510	0.798	0.403	(3.46)***	0.446	(3.88)***
	5	0.570	0.933	0.623	0.559	1.080	0.510	(2.77)***	0.589	(2.60)**
	Average	0.365	0.496	0.504	0.441	0.633	0.268	(4.36)***	0.303	(4.63)***

Panel B: First on $BVOL$ then on $IVOL$

		$IVOL$								
		1	2	3	4	5	High-Low		FF5	
$BVOL$	1	0.228	0.275	0.295	0.349	0.318	0.089	(1.21)	-0.010	(-0.13)
	2	0.286	0.241	0.346	0.353	0.352	0.066	(0.71)	-0.028	(-0.27)
	3	0.294	0.456	0.683	0.516	0.726	0.432	(4.17)***	0.244	(1.85)*
	4	0.266	0.425	0.528	0.477	0.889	0.622	(6.00)***	0.604	(5.56)***
	5	0.752	0.711	0.800	0.784	0.798	0.046	(0.40)	0.099	(0.69)
	Average	0.365	0.422	0.526	0.496	0.616	0.251	(4.98)***	0.182	(3.34)***

Table 7: Double Sorting: Extended Holding Periods

Similar to Table 6, at the end of each month, we first sort anomaly portfolios into quintiles based on idiosyncratic volatility (beta volatility) and then within each quintile, we further sort the anomalies into quintiles based on beta volatility (idiosyncratic volatility), which results in $5 \times 5 = 25$ portfolios. We report the equal-weighted average returns and the Fama-French five-factor model (FF5) risk-adjusted returns of the high-low portfolios for 3- and 6-month holding periods, respectively. We follow Jegadeesh and Titman (1993) to form portfolios with longer holding periods. Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is denoted by ***, **, and *, respectively. There are a total of 207 anomalies included in the regression. The sample period is from February 1948 to December 2022.

	Panel A: Control for $IVOL$				**Panel B: Control for** $BVOL$			
	$BVOL$				$IVOL$			
	3 months		6 months		3 months		6 months	
Rank	High-Low	FF5	High-Low	FF5	High-Low	FF5	High-Low	FF5
1	0.198	0.165	0.230	0.208	0.055	-0.071	0.085	-0.047
	(5.37)***	(4.18)***	(6.51)***	(5.24)***	(0.74)	(-0.89)	(1.18)	(-0.62)
2	0.155	0.158	0.148	0.143	0.067	-0.043	0.054	-0.056
	(2.96)***	(2.41)**	(3.03)***	(2.42)**	(0.75)	(-0.44)	(0.61)	(-0.58)
3	0.163	0.214	0.164	0.195	0.428	0.252	0.426	0.284
	(2.33)**	(2.47)**	(2.65)***	(2.79)***	(4.67)***	(2.03)**	(4.93)***	(2.40)**
4	0.357	0.397	0.397	0.457	0.594	0.593	0.577	0.533
	(3.44)***	(3.72)***	(4.02)***	(4.57)***	(6.46)***	(5.91)***	(6.59)***	(5.44)***
5	0.595	0.738	0.586	0.738	0.061	0.094	-0.003	0.039
	(3.42)***	(3.71)***	(3.42)***	(3.70)***	(0.64)	(0.87)	(-0.03)	(0.39)

Table 8: Beta Volatility and Anomaly Longevity

Panel regression results based on the following basic model:

$$CRET^a_{t+1,t+n} = b_0 + b_1\overline{BETA}^a_{t-m,t-1} + b_2\overline{BVOL}^a_{t-m,t-1} + b_3\overline{IVOL}^a_{t-m,t-1} + b_4 CRET^a_{t-m,t-1} + b_5(\overline{BVOL} \times CRET)^a_{t-m,t-1} + b_6(\overline{IVOL} \times CRET)^a_{t-m,t-1} + \gamma_a + \lambda_t + \epsilon^a_t$$

where $CRET^a_{t+1,t+n}$ and $CRET^a_{t-m,t-1}$are the cumulative return of anomaly a, n (m) month after (before) month t; the holding period n ranges from 3 to 36 months; the measuring period for past returns m is either 36 or 60 months. $\overline{BETA}^a_{t-m,t-1}$, $\overline{BVOL}^a_{t-m,t-1}$, and $\overline{IVOL}^a_{t-m,t-1}$ are the average market $BETA$, $BVOL$, and $IVOL$ for anomaly a in either the 36- or 60-month window prior to the formation month t; lastly,$(\overline{BVOL} \times CRET)^a_{t-m,t-1}$ and $(\overline{IVOL} \times CRET)^a_{t-m,t-1}$ represent the interaction terms between $\overline{BVOL}^a_{t-m,t-1}$ and $\overline{IVOL}^a_{t-m,t-1}$ with the past cumulative returns $CRET^a_{t-m,t-1}$. We control for time and cross-sectional fixed effects (λ_t and γ_a). The sample period is from February 1948 to December 2022.

Holding Months	3	6	9	12	36	3	6	9	12	36
	Measurement Period = 36 Months					Measurement Period = 60 Months				
$\overline{BETA}$	0.005	0.006	0.001	-0.004	-0.073	-0.002	-0.008	-0.018	-0.029	-0.062
	(0.58)	(0.40)	(0.03)	(-0.15)	(-1.49)	(-0.27)	(-0.53)	(-0.84)	(-1.09)	(-1.13)
$\overline{BVOL}(\times 100)$	0.049	0.118	0.193	0.270	0.315	0.102	0.153	0.243	0.376	0.164
	(0.45)	(0.60)	(0.71)	(0.79)	(0.40)	(0.77)	(0.72)	(0.86)	(1.04)	(0.20)
$\overline{IVOL}(\times 100)$	0.014	0.043	0.081	0.088	0.179	-0.012	-0.002	0.049	0.049	0.255
	(0.22)	(0.38)	(0.53)	(0.47)	(0.46)	(-0.14)	(-0.02)	(0.29)	(0.24)	(0.58)
$CRET$	-0.014	-0.034	-0.050	-0.068	-0.090	-0.002	-0.010	-0.023	-0.037	-0.110
	(-3.11)***	(-4.08)***	(-5.02)***	(-5.53)***	(-3.03)***	(-0.84)	(-2.12)**	(-3.46)***	(-4.48)***	(-5.99)***
$\overline{BVOL} \times CRET$	0.003	0.007	0.010	0.012	0.031	0.001	0.004	0.007	0.008	0.017
	(2.92)***	(3.67)***	(4.01)***	(4.15)***	(5.10)***	(1.22)	(2.90)***	(4.24)***	(4.08)***	(4.00)***
$\overline{IVOL} \times CRET$	-0.002	-0.004	-0.006	-0.006	-0.018	-0.001	-0.002	-0.004	-0.004	-0.003
	(-1.67)*	(-1.99)**	(-2.09)**	(-1.88)*	(-2.53)**	(-0.72)	(-1.71)*	(-2.35)**	(-1.72)*	(-0.65)

Table 9: Firm Level Evidence: Anomaly Enhancement by Beta Volatility

This table reports the Fama-MacBeth regression results of individual stock returns in month $t+1$ on firm characteristics and a composite anomaly index in month t using the following model:

$$RET_{it+1} = a_t + b_t BETA_{it} + c_t BVOL_{it} + d_t MPS_{it} + g_t(MPS \times BVOL_{it}) + h_t^{'} X_t + \epsilon_{it+1},$$

where RET_{it+1} is the return on stock i in month $t+1$; MPS represent the mispricing scores calculated by Stambaugh et al. (2015) minus 50, so that $MPS > 0$ indicates overpricing and $MPS < 0$ underpricing. $BETA$ and $BVOL$ are the estimated market beta and beta volatility for stock i in month t, respectively; $MPS \times BVOL$ is the interaction term between mispricing and beta uncertainty; X_{it} is a vector of other control variables for stock i in month t including LSZ_{it}, LBM_{it}, RET_{it}, AMD_{it} (the stock liquidity measure of Amihud (2002)), and $IVOL_{it}$. To match the ranking nature of MPS_{it} and help interpret the coefficients of the interaction terms, the decile ranks of $BVOL$ and $IVOL$ are used in the regression. We divide the sample into subsamples with overvalued stocks ($MPS > 0$) and undervalued stocks ($MPS < 0$), and report the regressions in Panels A and B, respectively. Newey and West (1987) robust t-statistics with 6 lagged terms are in parentheses and significance at the 1%, 5%, and 10% levels is denoted by ***, **, and *, respectively. Given the availability of MPS by Stambaugh et al. (2015), the firm level data sample period is from July 1965 to December 2016.

	Panel A: Overvalued Sample ($MPS > 0$)			**Panel B: Undervalued Sample** ($MPS < 0$)		
Intercept	0.027	0.027	0.027	0.020	0.029	0.021
	(5.80)***	(5.79)***	(5.79)***	(4.33)***	(5.15)***	(4.54)***
LSZ	-0.133	-0.136	-0.148	-0.104	-0.105	-0.106
	(-4.14)***	(-4.15)***	(-4.91)***	(-3.19)***	(-3.34)***	(-3.50)***
LBM	0.247	0.243	0.243	0.178	0.179	0.182
	(2.10)**	(2.07)**	(2.09)**	(1.87)*	(1.87)*	(1.90)*
AMD	-0.084	-0.086	-0.087	0.033	0.028	0.032
	(-1.42)	(-1.45)	(-1.49)	(0.84)	(0.73)	(0.79)
RET_{-1}	-0.050	-0.050	-0.052	-0.053	-0.054	-0.054
	(-6.88)***	(-6.98)***	(-7.19)***	(-9.64)***	(-9.67)***	(-9.69)***
MPS	-0.023	0.002	0.012	-0.017	-0.018	-0.015
	(-5.36)***	(0.29)	(1.28)	(-3.08)***	(-2.89)***	(-2.02)**
$BETA$	0.268	0.281	0.386	0.406	0.410	0.430
	(1.12)	(1.16)	(1.78)*	(2.00)**	(2.02)**	(2.28)**
$BVOL$	-0.029	0.021	0.013	-0.016	-0.010	-0.005
	(-2.07)**	(1.16)	(0.73)	(-1.92)*	(-0.95)	(-0.33)
$MPS \times BVOL$		-0.004	-0.004		0.000	0.001
		(-3.10)***	(-2.34)**		(0.22)	(0.74)
$IVOL$			-0.014			-0.022
			(-0.53)			(-0.92)
$MPS \times IVOL$			-0.002			-0.001
			(-0.79)			(-0.86)

Table 10: Beta Uncertainty and Arbitrage Activities

Fama-MacBeth cross-sectional regression results for:

$$SI_{it+1} = a_t + b_t BVOL_{it} + c_t IVOL_{it} + d_t^{'} X_{it} + \epsilon_{it+1},$$

where dependable variable SI_{it+1} (SI_{it}) is the short interest in month $t+1$ (t) for stock i. $BVOL_{it}$ and $IVOL_{it}$ are the decile ranks based on the estimated beta uncertainty and idiosyncratic volatility for stock i in month t, respectively; X_{it} is a vector of other firm characteristic variables for stock i in month t including LSZ_{it}, LBM_{it}, the past six month of returns $RET_{it(-6,-1)}$, and the percentage institutional ownership IOR_{it}. Panels A and B report the results for the contemporaneous and future arbitrage activities SI, respectively. Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is denoted by ***, **, and *, respectively. The sample period is from Jan 1973 to December 2022.

	Panel A: SI_{it}			**Panel B:** SI_{it+1}		
Intercept	0.009	0.011	-0.008	0.009	0.011	-0.008
	(10.81)***	(8.80)***	(-4.07)***	(10.65)***	(8.66)***	(-4.04)***
LSZ		-0.006	-0.006		-0.006	-0.006
		(-7.49)***	(-7.66)***		(-7.88)***	(-8.00)***
$LBM \times 100$		-0.006	0.3017		-0.003	0.3014
		(-0.11)	(4.67)***		(-0.05)	(4.68)***
$RET_{-6,-1}$		-0.003	-0.003		-0.003	-0.004
		(-2.65)***	(-2.74)***		(-3.01)***	(-3.17)***
$AMD \times 100$			-0.085			-0.097
			(-9.73)***			(-8.91)***
IOR			0.027			0.027
			(7.26)***			(7.14)***
$BETA$			0.002			0.002
			(3.77)***			(3.47)***
$BVOL \times 100$	-0.032	-0.040	-0.049	-0.031	-0.039	-0.049
	(-4.44)***	(-4.91)***	(-4.96)***	(-4.21)***	(-4.74)***	(-4.82)***
$IVOL \times 100$	0.208	0.177	0.197	0.210	0.181	0.201
	(9.56)***	(9.03)***	(8.74)***	(9.73)***	(9.18)***	(8.87)***

Table 11: Beta Uncertainty and Arbitrage Activities Proxied by Co-Momentum

The table reports the time series regression results based on this model:

$$CoMOM_{t+1} = a_t + b_t\overline{BVOL}_t + c_t\overline{IVOL}_t + d_t^{'}X_t + \epsilon_{t+1},$$

where dependant variable $CoMOM_{t+1}$ ($CoMOM_t$) is the comomentum measure, a proxy for arbitrage activity introduced by Lou and Polk (2022), in month $t+1$ (t). To obtain $CoMOM$, we follow Lou and Polk (2022) by adjusting each stock's daily return for its corresponding Fama-French 38 industry daily return and then by the Fama-French three-factor model each month. Stocks are sorted by the past six month of returns $RET_{-6,-1}$. The average pairwise Pearson correlation for loser and winner decile are then calculated each month based on the daily adjusted returns. $CoMOM$ is the average of the two average pairwise Pearson correlations for the loser and winner decile portfolios. Similar to $CoMOM$, we take $\overline{BVOL}_t$ and $\overline{IVOL}_t$ as the average estimated beta uncertainty and idiosyncratic volatility in month t for both winner and loser portfolios. X_t is a vector of other average firm characteristic variables for both winner and loser portfolios in month t, namely $\overline{BETA}_t$, $\overline{LSZ}_t$, and $\overline{TRNOV}_t$. Panels A and B report the results for the contemporaneous and future arbitrage activities $CoMOM$, respectively. Newey and West (1987) robust t-statistics are in parentheses and significance at the 1%, 5%, and 10% levels is denoted by ***, **, and *, respectively. The sample period is from Jan 1965 to December 2022.

	Panel A: $CoMOM_t$		**Panel B:** $CoMOM_{t+1}$	
Intercept	0.007	0.011	0.008	0.016
	(1.16)	(1.50)	(1.28)	(2.19)**
$\overline{BETA}$		0.003		0.001
		(0.79)		(0.33)
$\overline{BVOL}$	-0.024	-0.017	-0.025	-0.017
	(-3.26)***	(-2.54)**	(-3.30)***	(-2.67)***
$\overline{IVOL}$	0.144	0.003	0.143	-0.025
	(2.80)***	(0.08)	(2.60)***	(-0.81)
$\overline{LSZ}$		0.073		0.070
		(4.58)***		(4.29)***
$\overline{TRNOV}$		0.099		0.151
		(1.18)		(1.93)*

Michael Lee-Chin & Family Institute for Strategic Business Studies

Working Paper Series in Strategic Business Valuation

This working paper series presents original contributions focused on the theme of creation and measurement of value in business enterprises and organizations.

DeGroote School of Business at McMaster University
1280 Main Street West
Hamilton, Ontario, L8S 4M4