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Optimal Portfolio With Options*

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Abstract

We propose a parsimonious framework to analyze the optimal asset allocation problem for a mean–variance investor who incorporates options into the portfolio. Building on the key insight that risk-neutral volatility often differs from physical volatility, we derive closed-form expressions for the optimal portfolio weights with options. The resulting economic gains are substantial and remain robust even when solvency constraints and transaction costs are considered. We then take the model to the data and show that the framework delivers superior out-of-sample performance for an investor trading the S&P 500 index and its options. Our approach underscores the importance of accounting for differences between physical and risk-neutral volatilities when constructing an optimal portfolio that includes options.

Keywords: Option Investment, Portfolio Optimization, Variance Risk Premium, Investors' Utility Gains

JEL classification: G11, G12, G13.

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1 Introduction

Since the seminal work of Black and Scholes (1973), the options market has witnessed a substantial increase in trading volume by both institutional and retail investors. While most academic studies have concentrated on pricing and hedging options through the development of option pricing models, a growing body of literature has examined options as investment assets in their own right. The central intuition in this stream of research stems from the empirical observation that options are not redundant due to various market frictions, in contrast to the core premise of the Black–Scholes model. In theory, options should add no value to an investor's portfolio if they can be replicated using the underlying stock and the risk-free asset. Empirically, however, the options market is incomplete. Key sources of incompleteness extensively studied in the literature include stochastic volatility (Heston, 1993), jumps in stock returns (Pan, 2002), discrete trading and transaction costs (Leland, 1985), and borrowing constraints (Broadie, Cvitanic, and Soner, 1998), among others.

Given these market frictions, researchers have focused on deriving optimal trading strategies involving options and evaluating their performance. For instance, Liu and Pan (2003) show that trading derivatives can significantly improve portfolio outcomes when the underlying stock follows a stochastic process with stochastic volatility and price jumps. Goetzmann et al. (2002) demonstrate that the nonlinear payoffs of options can be exploited to enhance active investment performance. Alternatively, Faias and Santa-Clara (2017) propose a method to optimize option portfolios using a myopic objective function, showing that option portfolios can generate substantial utility gains beyond those achievable with underlying stocks alone.[1]

On the one hand, sophisticated option pricing models featuring rich dynamics of the underlying stock price process have delivered promising empirical performance in explaining

[1]Another strand of literature investigates option investment through cross-sectional long–short equity option portfolios; see, for example, Goyal and Saretto (2009), Cao and Han (2013), Vasquez (2017), Zhan et al. (2022), and Bali et al. (2023).

irregularities observed in the options market (e.g., the implied volatility smile). These models could, in principle, also be used to construct optimal trading strategies involving options. On the other hand, overly complex models often suffer from overfitting and estimation risk, leading to poor out-of-sample performance. Importantly, many such models were designed primarily to price options accurately, rather than to optimize portfolio construction. To address this issue, Faias and Santa-Clara (2017) adopt an extreme approach by relying solely on the empirical distribution of stock returns to simulate optimal option portfolios. In this paper, we propose a less extreme yet dramatically simplifying approach to demonstrate how an optimal portfolio involving options can be constructed and to evaluate its empirical performance in the S&P 500 index options market.

Specifically, rather than assuming complex dynamics for the underlying stock price process, we focus on a first-order empirical observation: volatility under the risk-neutral measure differs from volatility under the physical measure, contrary to the Black–Scholes model. This observation is most firmly established in the S&P 500 index options market, as documented in the extensive literature on the Variance Risk Premium (VRP) (Carr and Wu, 2009; Bollerslev, Tauchen, and Zhou, 2009). Our approach takes this fact as given, without attempting to explain why the two volatilities differ. We assume instead that stock returns are lognormally distributed under both the physical and risk-neutral measures, with distinct drift and volatility parameters. This assumption preserves a parsimonious analytical structure while reducing the risk of model misspecification and estimation error.

Under this framework, we derive closed-form expressions for the optimal portfolio weights of a mean–variance investor who can allocate between the underlying stock and options (both calls and puts). Because our objective is to construct and test a practically implementable portfolio, we first show theoretically that using only a small number of options can deliver performance close to that of the fully optimal portfolio that includes options across the entire strike spectrum. Moreover, our findings remain robust when solvency constraints and transaction costs are introduced. In particular, once solvency constraints are imposed, we demonstrate that trading options with implied volatility lower than physical volatility—namely, out-of-the-money (OTM) calls or in-the-money (ITM) puts—can still generate superior utility gains. This result contrasts sharply with much of the existing literature, which has emphasized profits from writing OTM puts, a strategy that often exposes portfolios to insolvency risk.

We then take our theoretical model to the data and apply it to construct an optimal portfolio using S&P 500 index and its options from January 1996 to August 2023. With only a minimal set of parameters to calibrate—specifically, the first two moments of the stock return distribution under the physical and risk-neutral measures—our framework is straightforward to implement empirically. The findings largely support the theoretical prediction that including options in a portfolio enhances utility gains for a mean–variance investor, even after accounting for solvency constraints and transaction costs. The gains are more pronounced as the investor's relative risk aversion increases. Thus, our framework offers a parsimonious setting with few parametric assumptions, yielding both theoretical and empirical improvements in optimal portfolio construction with options.

Our paper is closely related to a strand of literature that uses options to improve portfolio performance. The most closely related work is by Haugh and Lo (2001), who study equivalent buy-and-hold portfolios with options to approximate specific dynamic trading strategies under CRRA and CARA utilities. Other contributions include Carr and Madan (2001), who show that heterogeneity in preferences or beliefs induces investors to hold derivatives, and Goetzmann et al. (2002), who suggest that static rules involving two or more options—or even a continuum of derivative contracts—can maximize the Sharpe ratio. Moreover, Vanden (2004) argues that option portfolios act as additional factors in the cross-section of stock returns when agents face nonnegative wealth constraints; Basak and Chabakauri (2010) solve a dynamic mean–variance portfolio problem and derive a time-consistent optimal policy; and

Guasoni, Huberman, and Wang (2011) develop a performance-maximizing strategy using derivatives. We contribute to this literature by proposing a parsimonious semi-parametric framework that is easily implementable in practice and demonstrates how including options can improve out-of-sample performance.

The remainder of the paper is organized as follows. Section 2 presents the analytical results for option portfolios with finite holding periods. Section 3 extends the analysis to incorporate realistic economic constraints. Section 4 reports the empirical results using S&P 500 index options data, and Section 5 concludes.

2 Portfolio Optimization with Options

In this section, we introduce the basic setup of our theoretical model and present its implications. The main problem we consider is a simple optimization problem to maximize the mean-variance utility of a portfolio consisting of a stock and a risk-free asset, where the physical density of the stock return differs from the risk-neutral density for both mean and variance. In this case, we show that including options in the portfolio can result in a dramatic increase in the optimal Sharpe ratio.

2.1 Model Setup

Given a security with an initial value of $S_0 = 1$ and a payoff of S_T at time T, our objective is to determine the optimal function of S_T that gives us the highest expected utility. To be more specific, we would like to find a security with payoff $V_T = h(S_T)$ that has a current price of W_0 and maximizes $E[U(V_T)]$ for a given utility function $U(\cdot)$. The optimal choice of h depends on (1) the risk-neutral density of S_T, $f_Q(S_T)$, (2) the physical density of S_T, $f_P(S_T)$, and (3) the utility function $U(V_T)$, which we assume to be the classical mean-variance utility.

Let

$$g(S_T) = \frac{f_Q(S_T)}{f_P(S_T)} \tag{1}$$

be the ratio of the risk-neutral to the physical density of S_T, so that $m_k = e^{-rk} g(S_k)$ is a pricing kernel in the sense that the current price V_0 of any security with payoff $V_k = h(S_k)$ is $V_0 = E[m_k h(S_k)]$, where it is straightforward to check that $E[g(S_T)] = 1$. We are interested in finding a payoff, which is a function of S_T, i.e., $V_T = h(S_T)$ with an initial value of $V_0 = 1$, such that the Sharpe ratio of this payoff is maximized.[2] The Sharpe ratio of V_T is defined as

$$\theta_V = \frac{\mu_V}{\sqrt{\mathrm{Var}[V_T]}}, \tag{2}$$

where $\mu_V = E[V_T] - e^{rT}$ is the expected excess return of V_T. As given in Goetzmann et al. (2002) and Goetzmann et al. (2007), the solution to this problem is

$$V_T - e^{rT} = \mu_V \left[1 + \frac{1 - g(S_T)}{\mathrm{Var}[g(S_T)]} \right], \tag{3}$$

and the squared Sharpe ratio of V_T is

$$\theta_*^2 = \mathrm{Var}[g(S_T)]. \tag{4}$$

This can be shown to be greater than the squared Sharpe ratio of the stock itself, which is given by

$$\theta_S^2 = \frac{(E[S_T] - e^{rT})^2}{\mathrm{Var}[S_T]}. \tag{5}$$

It can also be easily verified that $V_0 = PV(V_T) = E[e^{-rT} g(S_T) V_T] = 1$. To see that, we have

$$PV(V_T - e^{rT}) = \mu_V E \left[e^{-rT} g(S_T) \left[1 + \frac{1 - g(S_T)}{\mathrm{Var}[g(S_T)]} \right] \right]$$

[2]A general solution to this kind of problem was given by Cox and Huang (1989) and Haugh and Lo (2001), where they consider the CRRA and CARA class of utility functions.

$$\Rightarrow \quad PV(V_T) - 1 = \mu_V e^{-rT}\left\{1 - \frac{E[g(S_T)[g(S_T)-1]]}{\mathrm{Var}[g(S_T)]}\right\}$$

$$\Rightarrow \quad PV(V_T) - 1 = 0. \tag{6}$$

Note that the Sharpe ratio of V_T is independent of scaling, so we can write

$$V_T - e^{rT} = c\left[1 + \frac{1 - g(S_T)}{\mathrm{Var}[g(S_T)]}\right] \tag{7}$$

for some constant $c > 0$. Therefore, the mean-variance optimal portfolio problem becomes a search for the portfolio that maximizes the utility function

$$U(V_T) = E[V_T - e^{rT}] - \frac{\gamma}{2}\mathrm{Var}[V_T], \tag{8}$$

where γ is the risk aversion coefficient. In other words, we need to find the value of c that maximizes the above mean-variance utility, and this can easily be shown to be

$$c^* = \frac{\theta_*^2}{\gamma}. \tag{9}$$

This leads to the optimal V_T that maximizes (8) given by

$$V_T^* - e^{rT} = \frac{\theta_*^2}{\gamma}\left[1 + \frac{1 - g(S_T)}{\mathrm{Var}[g(S_T)]}\right] = \frac{1}{\gamma}\left[\mathrm{Var}[g(S_T)] + 1 - g(S_T)\right]. \tag{10}$$

with the optimal utility

$$U(V_T^*) = \frac{\theta_*^2}{2\gamma}, \tag{11}$$

and this is greater than the mean-variance utility obtained from the optimal portfolio with stock and risk-free asset, which is $\theta_S^2/(2\gamma)$.

In order to implement the optimal portfolio V_T^*, we now need to make assumptions on the physical and risk-neutral density of the terminal stock price S_T, so that we can compute

$g(S_T)$ in equation (1). We assume under the physical measure, we have

$$\ln(S_T) - \ln(S_0) \sim N(T\tilde{\mu}, T\tilde{\sigma}^2), \tag{12}$$

and under the risk-neutral measure, we have

$$\ln(S_T) - \ln(S_0) \sim N\left(T\left(r - \frac{\sigma^2}{2}\right), T\sigma^2\right). \tag{13}$$

Note that under this assumption of risk-neutral measure for S_T, the Black-Scholes formula holds for European options. Although the annualized mean of the log stock return being $\tilde{\mu}$ and $r - \frac{\sigma^2}{2}$ is a typical result of risk-neutral pricing, we further allow the annualized variance of the log stock return to differ between the two measures as $\tilde{\sigma}^2$ and σ^2 under the physical and risk-neutral measure, respectively.

Haugh and Lo (2001) provides a couple of examples of stock price process under which $\tilde{\sigma} \neq \sigma$, but the Black-Scholes formula still holds as the risk-neutral distribution of stock returns is log-normal. Explicit examples include a trending Ornstein-Uhlenbeck process and a bivariate linear diffusion process with mean-reverting stochastic drift. In general, predictability of stock returns does not affect the validity of the Black-Scholes formula, but it can affect the volatility of the stock returns over a finite holding period (Lo and Wang, 1995; Jeon, Kan, and Li, 2025). Empirically, especially for the index options, there exists a rich literature on the variance risk premium (VRP) which provides ample evidence to believe that $\tilde{\sigma} \neq \sigma$, and most likely $\tilde{\sigma} < \sigma$.[3]

Without loss of generality, we first assume $S_0 = 1$. Then, the density of S_T under the physical and risk-neutral measures are given by

$$f_P(S_T) = \frac{1}{S_T\sqrt{2\pi}\sqrt{T}\tilde{\sigma}} \exp\left(-\frac{[\ln(S_T) - T\tilde{\mu}]^2}{2T\tilde{\sigma}^2}\right), \tag{14}$$

[3]For seminal works, see Bakshi and Kapadia (2003), Bollerslev, Tauchen, and Zhou (2009), and Carr and Wu (2009).

$$f_Q(S_T) = \frac{1}{S_T\sqrt{2\pi}\sqrt{T}\sigma} \exp\left(- \frac{\left[\ln(S_T) - T\left(r - \frac{\sigma^2}{2}\right)\right]^2}{2T\sigma^2} \right). \tag{15}$$

This gives us the functional form of the $g(S_T)$ as

$$g(S_T) = \frac{f_Q(S_T)}{f_P(S_T)} = \frac{\tilde{\sigma}}{\sigma} \exp\left(\alpha \ln(S_T)^2 + \beta \ln(S_T) + \lambda\right), \tag{16}$$

where

$$\alpha = \frac{1}{2T}\left(\frac{1}{\tilde{\sigma}^2} - \frac{1}{\sigma^2}\right), \tag{17}$$

$$\beta = \frac{r - \frac{\sigma^2}{2}}{\sigma^2} - \frac{\tilde{\mu}}{\tilde{\sigma}^2}, \tag{18}$$

$$\lambda = \frac{T\tilde{\mu}^2}{2\tilde{\sigma}^2} - \frac{T\left(r - \frac{\sigma^2}{2}\right)^2}{2\sigma^2}. \tag{19}$$

When $\tilde{\sigma} = \sigma$, we have $\alpha = 0$. Note that if $\tilde{\sigma} > \sigma$, we have $\alpha < 0$. In this case, $g(S_T)$ and therefore the state price density can be a decreasing function of S_T when S_T is large enough. However, when $\tilde{\sigma} < \sigma$, we have $\alpha > 0$, and the state price density can be an increasing function of S_T when S_T is large enough.

2.2 Model Implications

Given the optimal portfolio V_T^* derived in equation (10), we now study the implications of including options in the portfolio of a stock and risk-free asset. For the purpose of the numerical study, we assume that an investor optimizes her portfolio over one-period with one stock, risk-free asset, and possibly options written on the underlying stock. Without loss of generality, we focus on call options only.[4] The investors' risk aversion coefficient γ is set to 3. The holding period is chosen to be one month (1/12 year), and all options have

[4] The results of the put option are similar due to the put-call parity.

the same time to maturity as the holding period (i.e., one month) but with different strike prices. Without loss of generality, we assume that the initial stock price S_0 is \$1 and the stock price follows a geometric Brownian motion with a draft parameter of $\tilde{\mu} = 0.12$ and a standard deviation $\tilde{\sigma} = 0.16$ under the physical measure. The risk-free rate is set to $r = 0.04$. We then allow the standard deviation of the stock under the risk-neutral measure σ to vary between 0.14 and 0.20. The model parameters are calibrated on the basis of the S&P 500 index data from 1958. Without loss of generality, we assume that the dividend yield is zero.

It is easy to see that the Sharpe ratio of the optimal portfolio V_T^* will be quite different, indeed larger, from the static portfolio constructed at time 0 consisting of a stock and only the risk-free asset. This can be readily seen by plotting the ratio of risk-neutral to physical density, $g(S_T)$ as a function of S_T. Figure 1 plots $g(S_T)$ for different combinations of $\tilde{\sigma}$ and σ. When there are no differences between the physical and risk-neutral volatility ($\sigma = \tilde{\sigma}$), $g(S_T)$ is a monotonically decreasing function of S_T, a well established fact for the general pricing kernel as it represents the marginal utility. However, when the risk-neutral volatility is greater than the physical counterpart ($\sigma > \tilde{\sigma}$), we obtain a convex U-shaped function. Although it is inconsistent with standard economic theory, this phenomenon was empirically well-documented in previous non-parametric studies by Aït-Sahalia and Lo (2000), Jackwerth (2000), and Rosenberg and Engle (2002) using the S&P500 index options. Moreover, there is a large literature on the risk premium for variance of the S&P500 index options that documents the risk-neutral volatility consistently exceeding the physical volatility. Lastly, we also consider the case where risk-neutral volatility is smaller than the physical volatility ($\sigma < \tilde{\sigma}$), which results in a concave inverse U-shaped function.[5]

While it is a puzzling question why the empirical pricing kernel is U-shaped, we abstract from the fundamental economics behind it and rather focus on the application. That is, we

[5]Existing studies including Ziegler (2007), Bakshi, Madan, and Panayotov (2010), and Christoffersen, Heston, and Jacobs (2013) focus on providing economic intuitions and interpretations of the U-shaped pricing kernel. In comparison, we take the U-shaped pricing kernel as given, and focus on its implications on the optimal portfolio with options.

are interested in an optimal static portfolio problem involving options when the pricing kernel can be allowed to be either a convex or concave U-shaped function. Then it is immediately clear how the options can be particularly helpful when the risk-neutral volatility is different from the physical counterpart in Figure 1. For example, when $\sigma > \tilde{\sigma}$, consistent with empirical observation, the pricing kernel is increasing in both extremely negative and positive outcomes of the underlying stock price S_T. Therefore, including a security whose payoff resembles an Arrow-Debreu type of security in the extreme states will provide additional benefit to an investor's utility maximization problem. As options are precisely designed for this purpose, we expect to see significant improvements in maximum utility when options are included in an optimal portfolio when $\sigma \neq \tilde{\sigma}$.

The quadratic shape of the function $g(S_T)$ therefore naturally motivates the potential benefit of including the options in the static portfolio construction to mimic the optimal V_T^* as closely as possible. The choice of using mean-variance utility becomes handy in this case, because we can compute the required first and second moments of European option returns analytically in our setup. In fact, we derive a closed-form expression for all moments and cross-moments of European options and stocks for any maturity when σ is allowed to be different from $\tilde{\sigma}$. That is, we are not limiting ourselves to the case when the portfolio holding period is equal to the option maturity as typically assumed in the previous literature. Assuming the Black-Scholes option pricing formula holds, our closed-form expressions are also available when options mature beyond the portfolio holding period, where we use the expected Black-Scholes option price at the future date as the payoff in those cases.[6] The full derivations can be found in the Internet Appendix OA.1. Then, the optimal weights are given by standard solution to the mean-variance problem with many risk assets

[6] Rubinstein (1984) first derived the closed-form expression of European option return under the Black-Scholes model. More recently, Nawalkha and Zhuo (2022) provides a general change of measure framework for computing expected returns of derivative securities. We take a simpler approach of explicitly assuming log-normal distribution of stock returns under the physical and risk-neutral measures without specifying the underlying stock price process. This is sufficient for us to derive closed-form solutions for all higher and cross moments between stock and European options of various maturities.

$$w^{\star} = \frac{1}{\gamma}\Sigma^{-1}(\mu - r), \tag{20}$$

where Σ is a variance-covariance matrix of the returns of the holding period of stocks and options, μ is a vector of the expected returns of the holding period of stocks and options, γ is the coefficient of risk aversion and r is the risk-free rate.

One stream of existing literature that has studied the benefit of including options in the optimal portfolio often focuses on explicitly specifying the underlying stock price process and derive analytical implications. For example, Liu and Pan (2003) derives a dynamic trading strategy with options with the presence of stochastic volatility and jumps, and Eraker (2013) studies the optimal option trading strategy under parametric option pricing models. On the other hand, another stream of literature focuses on a completely non-parametric approach in understanding the benefit of including options. These studies include Guasoni, Huberman, and Wang (2011) who derives a theoretical maximal Sharpe ratio when one can trade options with any strike prices and also studies the problem when physical and risk-neutral volatilities can differ from each other, and Faias and Santa-Clara (2017) who takes a purely empirical approach to derive the optimal portfolio with options using a simulation based approach. Our approach falls in between, where we start from assuming the distribution of stock returns under the physical and risk-neutral measures, then focus on the performance of optimal options portfolio with a discrete small number of options only.

Equipped with the analytical expression of the optimal weights for a portfolio consisting of a stock, any number of call and put options, and a risk-free asset, we now plot how closely options portfolios can match the optimal payoff $V_T^*(S_T)$. Figure 2 plots the payoffs of different portfolios as a function of the terminal stock price S_T and compares it with $V_T^*(S_T)$. We consider four distinct cases: no options (stock only), and using 1, 2, and 3 call options, where the strike prices of the options were chosen by maximizing the mean-variance utility over strike price dimension, and the maturity of options is set to be equal to the holding period of portfolio (1 month). When $\sigma = \tilde{\sigma}$ in the top panel, we have not yet observed a visually compelling benefit of including options in the portfolio, as the optimal portfolio payoff V_T^* is close to a straight-line payoff of a portfolio with only a stock. However, in the two bottom panels where the risk-neutral volatility is different than the physical volatility, the optimal portfolio payoff V_T^* is highly non-linear, and thus using options helps tremendously in obtaining a shape closed to it. Even using a single call option delivers close to optimal payoff compared to the case of stock only.

Given that our analytical framework also allows for including options expiring beyond the portfolio holding period, we repeat the same analysis using the options maturing in more than 1 month. Figure 3 reports the result of using one call option in the portfolio that matures in 1, 2, 6, and 12 months. As the payoff of options maturing beyond 1 month will be given by the Black-Scholes option pricing formula at T, it will be a smooth increasing function of S_T rather than the hockey-stick shaped piece-wise linear payoff function of the hockey-stick of the standard European call option, thus potentially better suited to match the smooth U-shaped optimal payoff V_T^*. The findings in the middle and bottom panels are consistent with this intuition. In general, we find that the use of options that mature for more than 1 month matches the shape of V_T^* better than the use of an option that matures in 1 month.

Lastly and most importantly, we next plot the Sharpe Ratio of optimal portfolios as a function of risk-neutral volatility σ. Figure 4 plots the result by comparing the Sharpe Ratio of the optimal portfolio V_T^* to the same 4 cases that we considered above. First, as expected, we observe that the Sharpe Ratio of the optimal portfolio V_T^* is a concave U-shaped function, which means that the Sharpe Ratio is higher when the risk-neutral volatility σ is different from the physical volatility $\tilde{\sigma} = 16\%$ regardless of the direction. Because the return of the position of the stock is orthogonal to the risk-neutral volatility, the optimal portfolio including only the stock is a flat horizontal line that coincides with the optimal portfolio only when $\sigma = \tilde{\sigma}$. However, as we start including call options in the portfolio, the Sharpe Ratio of the optimal portfolio becomes dramatically closer to that of the optimal portfolio, a clear indication of the economic benefit of including options in the portfolio. Although having more options in the portfolio provides greater benefit as it should be, the marginal benefit of going from 2 to 3 options do not seem to be too high, indicating that even a smaller number of options can get us close to the optimal Sharpe Ratio. Figure 5 plots the same analysis using options that mature in 2 months instead. Consistent with the previous findings on the optimal payoff, we find that using options maturing in 2 months can get us closer to the optimal Sharpe Ratio with the same number of options in the portfolio.

In Table 1, we summarize the performance of optimal option portfolios for different values of $\sigma = 0.14, 0.16$, or 0.18 and using one, two or three call options in the portfolio. First, the benchmark case of an optimal portfolio is reported when options of all strike prices can be traded, where a Sharpe ratio of 0.210, 0.145, and 0.264 is obtained for $\sigma = 0.14, 0.16$, and 0.18, respectively. Consistent with the intuitions from the previous Figures, optimal portfolio improves Sharpe ratio as long as the risk-neutral volatility σ is different from the physical volatility $\tilde{\sigma}$. However, including stock only in the portfolio results in a slight underperformance compared to case $\sigma = \tilde{\sigma}$ and severely underperforms when $\sigma \neq \tilde{\sigma}$. This effect is more pronounced as the holding period increases to three-months in Panel B and one-year in Panel C. Surprisingly, including a more number of options beyond one does not dramatically increase the performance. Although increasing the number of options from one to two or three does improve the Sharpe ratio close to the optimal one, the improvement is rather marginal. For example, when $\sigma = 0.18$, Sharpe ratios of the optimal portfolio consisting of one, two, and three options are 0.240, 0.255, and 0.260, a moderate improvement.

We next report the performance of optimal portfolios when options are allowed to expire beyond the portfolio holding period of one-month. Table 2 reports the results considering the option maturities of two, three, and four-months while the portfolio is held for a month.

In general, we only observe a slight improvement by allowing options with longer maturity for the case of $\sigma > \tilde{\sigma}$ in Panel B using the one call option, but otherwise the difference is mostly muted.

In summary, the analytical exercises in this Section provide encouraging finding that including a very small, maybe just one, number of options can dramatically improve the performance of optimal stock portfolio when the risk-neutral volatility is different from the physical volatility. We find that a simple buy-and-hold strategy is shown to perform quite close to the theoretical optimal payoff, allowing options of all ranges of strike prices to be included in the portfolio. Subsequently, in the next section, we investigate further the performance of optimal option portfolios under various realistic economic constraints.

3 Optimal Option Portfolios with Economic Constraints

3.1 Solvency Constraint

Because we do not put any constraints on the sign of the weights, it is possible that the optimal portfolio weight indicates a short position in stocks or options. This could be particularly problematic in empirical applications, as naked short positions in the options can result in an insolvent portfolio. That is, the terminal value of the portfolio V_T can be negative in extreme cases. In this section, we discuss the constrained optimization problem that imposes the solvency constraint and show that our result is robust to it.

Formally, let k be the holding period of the portfolio and S_k be the price of the stock at $t = k$. For a given weight w, the value of the portfolio at $t = k$ is

$$V_k = w_0 e^{rk} + w_1 S_k + \sum_{i=1}^{n} \frac{w_{i+1}}{C_0(K_i)} C_k(K_i), \tag{21}$$

where $C_t(K_i)$ is the value of the call with exercise price K_i at time t. We would like to find the optimal portfolio with the constraint $V_k \geq 0$. Depending on the cases, the constraints can be either linear constraints that can be handled via quadratic programming or non-linear constraints that need to be handled via optimization. The Internet Appendix OA.4 contains all the details of how we handle each case and present the precise optimization problems that we solve.

Figure 6 reports the optimal Sharpe Ratio results in the same format as the unconstrained cases shown earlier. We have an interesting observation that using only 1 option significantly underperforms in the case when $\sigma > \tilde{\sigma}$ in the bottom panel, while using 3 options significantly underperforms in the case when $\sigma < \tilde{\sigma}$ in the middle panel. That is, it is no longer the case that using more number of options is strictly superior due to the solvency constraint that limits the amount of naked short positions in options. In general, using 2 options seems to provide the most stable result in this set of analyses.

Figure 7 plots the expected utility with $\gamma = 3$ of the optimal portfolios of stocks only and includes a call option against the optimal portfolio V_k^* with and without solvency constraint. As the risk-averse investor with $\gamma = 3$ is unlikely to take levered positions in the underlying stock, imposing the solvency constraint has no effect on the stock only portfolio. When a call option is included in the portfolio, the solvency constraint significantly reduces the expected utility of the portfolio with a call option in the region where the risk-neutral volatility σ is greater than the physical volatility $\tilde{\sigma}$. In contrast, virtually no loss in expected utility is observed when σ is lower than $\tilde{\sigma}$, because it is unlikely that it will be optimal to take short positions in the option in such a case. When we assume a risk-neutral investor with $\gamma = 1$, the findings become more pronounced. Figure 8 plots the case $\gamma = 1$. First, now even the portfolio with stock only suffers a severe drop in the expected utility under the solvency constraint, as they are not allowed to aggressively take leverage in the underlying stock. In this case, interestingly, the use of a call option can still provide the expected utility of the unconstrained stock position even when σ is greater than $\tilde{\sigma}$. This is because the use of a call option can effectively allow an investor to take optimal leverage that was not possible using the stock only under the solvency constraint. As previously, the solvency constraint does not affect the expected utility of the optimal portfolio with a call option when σ is lower than $\tilde{\sigma}$. These findings point to a potential benefit of including deep out-of-the-money call options or in-the-money put options with implied volatility smaller than the physical volatility.

Table 3 reports the expected mean-variance utilities of optimal option portfolios with and without the solvency constraint for various values of γ ranging from 1 to 10. Consistent with the Figures, we find that the solvency constraint reduces the expected utility of optimal option portfolios the most when $\sigma > \tilde{\sigma}$ in Panel B. Moreover, the reduction is more significant as the investor becomes more risk-taking with lower γ. However, the reduction is moderate or non-existent when the coefficient of risk aversion is greater than or equal to 2 and $\sigma \leq \tilde{\sigma}$. Overall, while the strict solvency constraint weakens the performance of the optimal portfolio as expected, the benefit of including options in the optimal portfolio is robust to the solvency constraint.

3.2 Bid-Ask Spread

It is also well-known that trading options incur larger transaction costs than trading underlying stocks. The bid-ask spreads of the option contracts are often substantially larger than the underlying stocks, which could potentially dilute the performance of the portfolio empirically. We therefore check whether the mean-variance optimal portfolio with options we derived survives the potentially high transaction cost in the form of bid-ask spread. In other words, for a set of bid-ask spreads, we assume that options are always bought at the ask price and sold at the bid price. When there is a n number of options, we need to solve 2^n number of individual optimization problems, each with different constraints on the sign of the weights of the n options, to determine the optimal portfolio. In other words, the optimal portfolio is obtained by selecting the best performing portfolio of the 2^n optimal portfolios considered.

Fig 9 plots the expected utility of an optimal portfolio with a call option under the assumption of bid-ask spread equal to 0%, 1% and 5% of the spot price. These are reasonable assumptions, as Kaeck, van Kervel, and Seeger (2022) reports the effective spread of the SPX index option of 1.69% in the 2014 sample while Muravyev and Pearson (2020) reports the average 2.2% effective spread in the cross-section of equity options. Given that our empirical study later focuses on the SPX index options, the assumption of 1% bid-ask spread is likely quite close to the actual transaction costs. Nevertheless, we find that even using 5% of bid-ask spread assumption leads to substantial improvement over the stock only portfolio while it still relatively performs well close to the optimal and 1% assumption cases. The loss in expected utility due to the 1% bid-ask spread assumption is rather minimal and does not alter our results both qualitatively and quantitatively. In general, our result is robust to a realistic assumption of bid-ask spread on portfolio of the stock and one call option.

4 Empirical Evidence

4.1 Data and Parameter Calibration

We collect S&P500 index options data including bid price, ask price, trading volume, open interest, dividend yield, implied volatility, expiration date, and strike price from OptionMetrics database. The sample period is January 1996 to August 2023. We also obtain the benchmark risk-free rate and the value of the SPX index from CRSP. We apply standard filters to remove options with zero trading volume and with bid price less than $1.25. For the purpose of empirical analysis, we focus on monthly SPX options that mature on the third Friday of each month. That is, we form a portfolio with a one-month holding period on the third Friday of each month, where the option position is to be held until the option expiration date next month. This gives us a total of 332 monthly observations.

Table 4 reports the summary statistics of the option data. First, we observe that trading volumes are heavily skewed towards in-the-money for call options and out-of-the-money for put options, while at-the-money options show the highest trading volume. Second, the implied volatilities exhibit the well-known "smirk" pattern that it is a decreasing function of the moneyness, while the left side of the curve (moneyness less than 1) has a steeper decline. Lastly, the average held-to-expiration return of call options is overall positive with the exception of deep in-the-money option with moneyness close to 1.05. As investors in general are willing to pay a premium to hedge against downside risk, consistent with the existing literature, the average put option returns are all extremely negative. Table **??** shows the increase in trading volume over time. From 1997 to 20222, the trading volume of both call and put options increased by roughly 20 times, while the puts always had a strictly greater trading volume than calls.

To construct an optimal option portfolio, we need to calibrate two parameters in the physical measure: the mean return μ and the physical volatility $\tilde{\sigma}$. It is extremely difficult to estimate precisely μ, especially with the short time series data we have. We therefore take a simplistic approach and use the average return of the S&P500 index in a rolling window of 360 months. We note that one benefit of constructing portfolios with options is that they are relatively less sensitive to the estimation risk associated with μ. In contrast, there are multiple promising methods to estimate $\tilde{\sigma}$. We use various methods and confirm that our findings are robust to the particular choice of the model. These include realized volatility of the last 22 trading days observation and forecast from the heterogeneous autoregressive model of realized volatility (HAR-RV) from Corsi (2009). For the risk-neutral measure, we directly use the risk-free rate r obtained from the CRSP and the implied volatility σ obtained from OptionMetrics.

4.2 Empirical Results

We conduct our empirical study with a simple and trackable case in which investors choose options with maturity equal to their investment holding period of one-month. As in the theoretical analysis from the previous sections, we assume that the risk-neutral volatility can be different from the physical volatility. As the main empirical question we try to address is whether the optimal portfolio including options can significantly outperform the optimal portfolio using the stock only, we exclusively rely on the out-of-sample performance to compare the performances. Indeed, it is not clear at all whether including options in the portfolio will lead to better performance out-of-sample as we only rely on a simplistic assumption of log-normally distributed stock returns under the physical and risk-neutral measures.

Specifically, by solving the optimization problem we explored in the theoretical section, we find optimal weights associated with the stock w_s^* and an option $w_o^*(K)$ for the given strike price K at the third-price of each month using the estimated parameters $r, \mu, \tilde{\sigma}$, and σ with different values of investors' risk-aversion γ. We assume that σ corresponds to the implied volatility of the option contract included in the portfolio. Then, the portfolio return r_{t+1}^P for month $t+1$ is given by

$$r_{t+1}^P = w_s^* \left(\frac{S_{t+1}}{S_t} - 1 \right) + w_o^*(K) \left(\frac{O_{t+1}(K)}{O_t(K)} - 1 \right) + (1 - w_s^* - w_o^*(K)) e^{r \Delta t}, \qquad (22)$$

where $O_{t+1}(K)$ is the terminal option payoff and $O_t(K)$ is the observed price of the option at time t given by the mid-price.

Furthermore, we ensure that we only consider practically implementable portfolios. That is, we impose a solvency constraint requiring that the portfolio value not fall below 0 at maturity. This restriction is critical for the empirical evaluation of the options portfolio because otherwise shorting out-of-the-money put options will likely deliver a sharp increase in performance. We emphasize that our approach is distinct from the known strategies, often known to be the source of hedge fund returns, that harvest insurance premiums from the OTM put options.

Figure 10 shows the out-of-sample (OOS) performance of the three optimal portfolios: i) stock only, ii) stock and one call option, and iii) stock and one put option, by plotting the OOS Sharp ratios as a function of the moneyness of the option contract used in the optimal portfolio. Each panel corresponds to the different value of investor's risk aversion γ where we consider cases of γ equal to 1, 2, 3, and 5. We first observe that the OOS Sharpe ratios of portfolios including options is an increasing function of the moneyness. In other words, by including either OTM call options or ITM put options, one can obtain an increased OOS performance of the portfolio. Second, the performance of the OOS of the portfolio with options strictly increases as the investors' risk aversion γ increases. In other words, an investor will be more likely to benefit from including options in their portfolio when they are more risk-averse. When $\gamma = 1$, an investor will need to include an option with large moneyness exceeding 1.03 to outperform holding a stock only in her portfolio, but can outperform with an option in virtually all cases when $\gamma = 5$. Lastly, using a put option seems to provide better performance in all cases than using a call option, perhaps due to the premium attached with the put options. Overall, OOS performance delivers a promising result that strongly supports our theoretical result that constructing an optimal mean-variance portfolio involving an option can be largely beneficial to an investor. In particular, we have shown that using an option with single strike price is sufficient to obtain such benefit.

We provide details of the optimized portfolios with options in Table 5. The table reports the performance of the optimal portfolio with one option (either call or put) with various choices of moneyness. For comparison, we include the optimal portfolio with stock only. According to Figure 10, both the OOS Sharpe ratio and utility are higher in most cases under the optimized portfolio with option than in those with stock only.

One of the main issues when implementing a trading strategy involving options is the relatively high transaction cost. Our approach minimizes the potential transaction costs involved because we only trade the option once in the portfolio formation, which is then held until maturity. Nevertheless, we next check whether our results survive the transaction cost by assuming that all buy orders are executed at the best ask price, and that all sell orders are executed at the best bid price instead. Figure 11 plots the result with both the solvency constraint and the transaction cost, thus being the closest as possible to the practically implementable portfolio. Not surprisingly, the overall performance is relatively lower than in Figure 10 where only a solvency constraint was imposed. In particular, the benefit of including an option in the portfolio seems to be not clearly present when an investor with risk-neutral with $\gamma = 1$. However, strict performance over the stock can still be obtained only when γ is greater than 1. Interestingly, the pattern of strict dominance of using a put option over a call option no longer holds. With the transaction cost considered, it often turns out that using a call option can be more beneficial. Although we assume a 100% bid-ask spread for the analysis, Muravyev and Pearson (2020) finds that the actual transaction costs of the trading options can be significantly lower, so the actual empirical performance might be somewhere between Figures 10 and 11.

Table 5 reports the characteristics of the optimal portfolio with the solvency constraint. While the performance metrics including out-of-sample Sharpe ratios and utilities are consistent with the Figures, it is informative to look at the average weights of the stock positions in the optimal portfolio. When the investor is close to risk-neutral with γ ranging from 1 to 3, an optimal portfolio is given by reducing the weight of the stock and taking a long position in OTM call options, or taking a levered stock position and hedge it with a long position in ATM put options. On the other hand, when the investor is risk-averse with γ equal to 5, the optimal portfolio increases the position of the stock while taking a small short position in the OTM call option to reduce the variance, which coincides with the investment philosophy of the covered call strategy. Table 6 reports the same result with full consideration of transaction cost. The results are qualitatively quite similar to the case with solvency constraint only, with a tendency to have more short positions in the options overall. This is consistent with the well-known notion that market makers overall take short positions in the index options, and hence taking short positions can help capturing the spread in favor of sellers. Thus, we conclude that our findings are largely robust to the inclusion of the transaction costs involved with the portfolio construction.

5 Conclusion

In contrast to the Black–Scholes model, it is well established that options are not redundant in practice due to the presence of various market frictions. A salient feature of the options market is that risk-neutral volatility often differs substantially from physical volatility. Building on this observation, we construct an analytical framework for the optimal mean–variance portfolio consisting of a stock and its options. Unlike many existing studies, our approach does not specify a detailed process for the underlying stock price. Instead, we assume a simple log-normal distribution of stock returns under both the physical and risk-neutral measures, allowing for potentially different volatility parameters, and show that including options can significantly enhance portfolio performance. Our results remain robust when solvency constraints and bid–ask spreads are incorporated. Applying the framework to S&P 500 index option data from 1996 to 2023, we find that it delivers superior out-of-sample performance. These findings contribute to the growing literature on options as investment assets and demonstrate that a parsimonious framework with minimal assumptions can yield substantial improvements in portfolios combining stocks and options. Moreover, our theoretical and empirical results provide insights into why options with certain levels of moneyness attract greater trading interest.

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Table 1
Performances of Optimal Option Portfolios

Panel A. One-month Holding Period													
		Optimal		Stock		One Call			Two Calls			Three Calls	
σ	0.14	0.16	0.18	0.16	0.14	0.16	0.18	0.14	0.16	0.18	0.14	0.16	0.18
w_s				1.027	−1.781	1.291	6.085	−2.557	1.447	13.344	−2.837	1.559	23.027
w_c^1					0.100	−0.009	−0.251	0.032	−0.017	−0.875	0.108	−0.023	−1.869
K_c^1					(0.988)	(1.001)	(0.971)	(1.007)	(0.963)	(0.926)	(0.948)	(0.942)	(0.900)
w_c^2								0.122	−0.002	−0.048	0.053	−0.005	−0.182
K_c^2								(0.962)	(1.037)	(1.016)	(0.982)	(0.998)	(0.966)
w_c^3											0.017	−0.001	−0.014
K_c^3											(1.015)	(1.059)	(1.050)
SR	0.210	0.145	0.264	0.144	0.209	0.145	0.240	0.211	0.145	0.255	0.211	0.145	0.260

Panel B. Three-months Holding Period													
		Optimal		Stock		One Call			Two Calls			Three Calls	
σ	0.14	0.16	0.18	0.16	0.14	0.16	0.18	0.14	0.16	0.18	0.14	0.16	0.18
w_s				0.998	−1.000	1.523	6.265	−1.650	1.869	15.586	−1.580	2.136	29.402
w_c^1					0.141	−0.028	−0.478	0.171	−0.062	−1.917	0.172	−0.087	−4.419
K_c^1					(0.960)	(1.002)	(0.938)	(0.915)	(0.937)	(0.863)	(0.920)	(0.900)	(0.820)
w_c^2								0.056	−0.006	−0.105	0.045	−0.018	−0.437
K_c^2								(0.985)	(1.064)	(0.999)	(0.995)	(0.995)	(0.919)
w_c^3											-0.0002	−0.002	−0.034
K_c^3											(1.186)	(1.104)	(1.046)
SR	0.280	0.254	0.371	0.247	0.277	0.253	0.347	0.278	0.254	0.362	0.279	0.254	0.367

Panel C. One-year Holding Period													
		Optimal		Stock		One Call			Two Calls			Three Calls	
σ	0.14	0.16	0.18	0.16	0.14	0.16	0.18	0.14	0.16	0.18	0.14	0.16	0.18
w_s				0.877	1.029	2.324	9.106	−0.476	3.634	31.072	−0.405	4.837	73.522
w_c^1					−0.002	−0.144	−1.489	0.308	−0.4	−8.091	0.291	−0.653	−22.904
K_c^1					(1.314)	(1.007)	(0.876)	(0.844)	(0.874)	(0.735)	(0.851)	(0.804)	(0.661)
w_c^2								−0.004	−0.029	−0.357	−0.005	−0.104	−1.9
K_c^2								(1.275)	(1.127)	(0.971)	(1.209)	(0.982)	(0.820)
w_c^3											0.0004	−0.009	−0.132
K_c^3											(1.410)	(1.209)	(1.040)
SR	0.490	0.533	0.708	0.477	0.487	0.526	0.658	0.490	0.531	0.690	0.491	0.532	0.700

This table reports the weights, strike prices, and Sharpe ratio (SR) of optimal option portfolios. We report the theoretically optimal Sharpe ratio assuming options with all ranges of strike prices can be traded first, then followed by the optimal stock only portfolio and including one, two, and three call options. In Panels A, B, and C, we report the result for the portfolio holding period of one-month, three-months, and one-year, respectively. All options have maturity equal to the holding period of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $\tilde{\sigma} = 0.16$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

Table 2
Performances of Optimal Portfolios with Options Maturing Beyond Holding-period

Panel A. $\sigma = \tilde{\sigma}$

	Optimal	Stock	One Call			Two Calls			Three Calls		
	K=1	K=1	K=2	K=3	K=4	K=2	K=3	k=4	K=2	K=3	K=4
w_s		1.027	1.474	1.610	1.733	1.830	2.116	2.386	2.160	2.615	3.057
w_c^1			−0.023	−0.038	−0.054	−0.058	−0.101	−0.150	−0.094	−0.172	−0.263
K_c^1			(0.996)	(0.991)	(0.987)	(0.934)	(0.915)	(0.899)	(0.892)	(0.863)	(0.840)
w_c^2						−0.004	−0.006	−0.008	−0.015	−0.026	−0.037
K_c^2						(1.059)	(1.071)	(1.080)	(0.994)	(0.989)	(0.985)
w_c^3									−0.001	−0.001	−0.001
K_c^3									(1.107)	(1.133)	(1.154)
SR	0.145	0.144	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145	0.145

Panel B. $\sigma > \tilde{\sigma}$

	Optimal	Stock	One Call			Two Calls			Three Calls		
	K=1	K=1	K=2	K=3	K=4	K=2	K=3	K=4	K=2	K=3	K=4
w_s		1.027	15.186	26.642	42.936	65.522	188.423	490.757	267.685	1512.975	81403.609
w_c^1			−1.254	−3.136	−6.477	−9.263	−37.248	−121.912	−50.785	−398.244	−30383.438
K_c^1			(0.932)	(0.899)	(0.867)	(0.860)	(0.809)	(0.761)	(0.810)	(0.742)	(0.635)
w_c^2						−0.056	−0.059	−0.061	−0.563	−1.138	−5.328
K_c^2						(1.059)	(1.102)	(1.147)	(0.947)	(0.923)	(0.862)
w_c^3									−0.009	−0.009	−0.013
K_c^3									(1.155)	(1.239)	(1.299)
SR	0.264	0.144	0.256	0.258	0.259	0.263	0.264	0.264	0.264	0.264	0.264

Panel C. $\sigma < \tilde{\sigma}$

	Optimal	Stock	One Call			Two Calls			Three Calls		
	K=1	K=1	K=2	K=3	K=4	K=2	K=3	K=4	K=2	K=3	K=4
w_s		1.027	−3.539	−5.062	−6.605	−3.235	−3.630	−4.095	−3.301	−0.440	−0.221
w_c^1			0.249	0.427	0.644	0.221	0.317	832.865	0.223	−0.625	−1.004
K_c^1			(0.977)	(0.966)	(0.957)	(0.984)	(1.010)	(1.054)	(0.981)	(0.868)	(0.897)
w_c^2						−0.001	−0.052	−832.544	0.005	0.383	0.755
K_c^2						(1.142)	(1.074)	(1.054)	(1.057)	(0.984)	(0.985)
w_c^3									−0.002	−0.009	−0.050
K_c^3									(1.122)	(1.122)	(1.097)
SR	0.212	0.144	0.211	0.211	0.210	0.212	0.212	0.212	0.212	0.212	0.212

This table reports the weights, strike prices, and Sharpe ratio (SR) of optimal option portfolios using options maturing beyond the portfolio holding-period of one-month. We report the theoretically optimal Sharpe ratio assuming options with all ranges of strike prices can be traded first, then followed by the optimal stock only portfolio and including one, two, and three call options. For each portfolio using options, we consider option maturity of two-months, three-months, and four-months (K=2, 3, and 4). The Panel A reports the result when $\sigma = \tilde{\sigma} = 0.16$, the Panel B reports the result when $\sigma = 0.18$ is greater than $\tilde{\sigma}$, and the Panel C reports the result when $\sigma = 0.14$ is smaller than $\tilde{\sigma}$. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $\tilde{\sigma} = 0.16$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

Table 3
Expected Utilities of Optimal Option Portfolios with Solvency Constraint

Panel A. $\sigma = \tilde{\sigma}$

		Stock			One Call			Two Calls			Three Calls		
Gamma	Optimal	NC	C	NC/C	NC	C	NC/C	NC	C	NC/C	NC	C	NC/C
1	0.0105	0.0103	0.0056	1.8386	0.0105	0.0103	1.0160	0.0105	0.0105	1.0017	0.0105	0.0105	1.0004
2	0.0053	0.0052	0.0045	1.1403	0.0053	0.0052	1.0160	0.0053	0.0053	1.0017	0.0053	0.0053	1.0004
3	0.0035	0.0034	0.0034	1.0007	0.0035	0.0035	1.0120	0.0035	0.0035	1.0017	0.0035	0.0035	1.0004
5	0.0021	0.0021	0.0021	1.0000	0.0021	0.0021	1.0000	0.0021	0.0021	1.0000	0.0021	0.0021	1.0000
10	0.0011	0.0010	0.0010	1.0000	0.0011	0.0011	1.0000	0.0011	0.0011	1.0000	0.0011	0.0011	1.0000

Panel B. $\sigma > \tilde{\sigma}$

		Stock			One Call			Two Calls			Three Calls		
Gamma	Optimal	NC	C	NC/C	NC	C	NC/C	NC	C	NC/C	NC	C	NC/C
1	0.0349	0.0103	0.0056	1.8386	0.0289	0.0103	2.7975	0.0326	0.0267	1.2194	0.0338	0.0274	1.2358
2	0.0174	0.0052	0.0045	1.1403	0.0145	0.0056	2.5851	0.0163	0.0144	1.1342	0.0169	0.0152	1.1111
3	0.0116	0.0034	0.0034	1.0007	0.0096	0.0050	1.9270	0.0109	0.0096	1.1288	0.0113	0.0104	1.0815
5	0.0070	0.0021	0.0021	1.0000	0.0058	0.0041	1.4196	0.0065	0.0058	1.1287	0.0068	0.0063	1.0685
10	0.0035	0.0010	0.0010	1.0000	0.0029	0.0027	1.0837	0.0033	0.0029	1.1287	0.0034	0.0032	1.0672

Panel C. $\sigma < \tilde{\sigma}$

		Stock			One Call			Two Calls			Three Calls		
Gamma	Optimal	NC	C	NC/C	NC	C	NC/C	NC	C	NC/C	NC	C	NC/C
1	0.0224	0.0103	0.0056	1.8386	0.0219	0.0219	1.0000	0.0223	0.0223	1.0000	0.0224	0.0224	1.0000
2	0.0112	0.0052	0.0045	1.1403	0.0109	0.0109	1.0000	0.0112	0.0112	1.0000	0.0112	0.0112	1.0000
3	0.0075	0.0034	0.0034	1.0007	0.0073	0.0073	1.0000	0.0074	0.0074	1.0000	0.0075	0.0075	1.0000
5	0.0045	0.0021	0.0021	1.0000	0.0044	0.0044	1.0000	0.0045	0.0045	1.0000	0.0045	0.0045	1.0000
10	0.0022	0.0010	0.0010	1.0000	0.0022	0.0022	1.0000	0.0022	0.0022	1.0000	0.0022	0.0022	1.0000

This table reports the expected mean-variance utility of optimal option portfolios for different value of risk aversion coefficient γ. We report the theoretically optimal expected utility assuming options with all ranges of strike prices can be traded first, then followed by the optimal stock only portfolio and including one, two, and three call options. Columns with label NC corresponds to the case when no constraint is imposed, columns with label C corresponds to the case with the solvency constraint, and the columns with label NC/C reports the ratio between the expected utilities of unconstrained and constrained cases. The Panel A reports the result when $\sigma = \tilde{\sigma} = 0.16$, the Panel B reports the result when $\sigma = 0.18$ is greater than $\tilde{\sigma}$, and the Panel C reports the result when $\sigma = 0.14$ is smaller than $\tilde{\sigma}$. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $\tilde{\sigma} = 0.16$, $r = 0.04$, $S_0 = 1$, and $\delta = 0$.

Table 4
Summary Statistics of the S&P500 Index Option Contracts

	Moneyness (K/S)				
	0.95	0.975	1	1.025	1.05
	Panel A: Calls				
Volume	263.127	554.599	5790.759	2444.434	4340.274
Bid-ask Spread	0.031	0.040	0.054	0.113	0.235
Implied Volatility	0.207	0.192	0.172	0.155	0.149
Average Return	0.070	0.083	0.125	0.224	−0.066
Standard Deviation	0.675	0.841	1.245	2.694	4.238
10th Percentile	−1.000	−1.000	−1.000	−1.000	−1.000
25th Percentile	−0.420	−0.748	−1.000	−1.000	−1.000
50th Percentile	0.116	0.075	−0.257	−1.000	−1.000
75th Percentile	0.564	0.744	1.068	0.532	−1.000
90th Percentile	0.894	1.182	1.945	3.016	1.151
	Panel B: Puts				
Volume	2554.003	2656.090	5120.491	621.373	205.792
Bid-ask spread	0.106	0.076	0.055	0.047	0.037
Implied Volatility	0.218	0.198	0.177	0.162	0.156
Average Return	−0.411	−0.288	−0.197	−0.155	−0.128
Standard Deviation	3.122	2.403	1.765	1.231	0.925
10th Percentile	−1.000	−1.000	−1.000	−1.000	−1.000
25th Percentile	−1.000	−1.000	−1.000	−1.000	−0.834
50th Percentile	−1.000	−1.000	−1.000	−0.613	−0.350
75th Percentile	−1.000	−1.000	−0.071	0.326	0.308
90th Percentile	−0.822	1.516	2.008	1.366	1.010

This table reports summary statistics of the S&P500 index option contracts with one-month maturity. The options are grouped together based on their moneyness defined as the strike price over the spot price of the S&P500 index. Return of each option is computed as the terminal payoff over the mid-price at the third Friday of each month. All data are from the OptionMetrics database. The sample period is from January 1996 to August 2023.

Table 5
Option Portfolio Performance with Solvency Constraints

	Stock	Call Option					Put Option				
		Moneyness (K/S)					Moneyness (K/S)				
		0.95	0.98	1	1.02	1.05	0.95	0.98	1	1.02	1.05
Panel A: Gamma=1											
w_o		0.102	0.024	0.014	0.012	0.004	0.004	0.003	0.016	0.087	0.301
$w_o\% > 0$		53.313	40.964	40.361	45.181	41.867	55.723	41.566	38.253	42.470	44.277
w_s	0.999	0.879	0.945	0.863	0.819	0.891	3.100	1.985	2.292	4.440	7.698
$w_s\% > 0$	100.000	98.795	99.096	96.687	94.277	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.138	0.063	0.079	0.106	0.136	0.154	0.068	0.084	0.113	0.155	0.178
OOSU	0.006	−0.001	0.003	0.006	0.009	0.009	−0.001	0.004	0.006	0.011	0.011
Panel B: Gamma=2											
w_o		0.024	−0.006	−0.004	0.000	0.000	−0.002	−0.006	−0.003	0.026	0.116
$w_o\% > 0$		42.169	32.530	32.530	37.952	36.446	44.277	33.133	31.325	36.747	40.060
w_s	0.991	0.960	0.978	0.920	0.874	0.914	1.575	0.961	1.082	2.081	3.577
$w_s\% > 0$	100.000	99.096	99.096	96.687	94.277	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.142	0.082	0.116	0.138	0.167	0.187	0.090	0.118	0.144	0.184	0.208
OOSU	0.005	0.001	0.003	0.005	0.006	0.007	0.001	0.003	0.005	0.008	0.008
Panel C: Gamma=3											
w_o		−0.003	−0.016	−0.010	−0.003	−0.002	−0.004	−0.008	−0.009	0.006	0.056
$w_o\% > 0$		36.747	26.506	26.807	33.434	32.530	37.952	25.904	25.301	32.831	37.651
w_s	0.967	0.987	0.988	0.937	0.888	0.914	1.043	0.620	0.681	1.303	2.222
$w_s\% > 0$	100.000	99.398	99.096	96.687	94.277	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.145	0.098	0.139	0.153	0.179	0.196	0.109	0.139	0.159	0.197	0.213
OOSU	0.004	0.001	0.003	0.004	0.005	0.006	0.002	0.003	0.004	0.006	0.007
Panel D: Gamma=5											
w_o		−0.024	−0.024	−0.014	−0.006	−0.002	−0.006	−0.010	−0.013	−0.009	0.010
$w_o\% > 0$		25.904	18.373	17.470	26.807	27.711	26.205	15.060	16.867	25.602	31.928
w_s	0.858	1.007	0.995	0.946	0.888	0.890	0.617	0.351	0.365	0.696	1.166
$w_s\% > 0$	100.000	99.398	99.096	96.687	94.277	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.129	0.126	0.160	0.166	0.184	0.192	0.135	0.158	0.170	0.200	0.204
OOSU	0.001	0.001	0.003	0.003	0.003	0.004	0.002	0.003	0.003	0.004	0.004

This table summarizes the performance of the optimal portfolio with options. As a comparison, we include the optimal portfolio with stock only. For each portfolio with options, we only include one option with the moneyness specified at the top of each column. The portfolio is rebalanced by the end of the third Friday of each month with the same moneyness over time. The reported statistics include the average portfolio weights for stocks and options, the proportion of positive weights over all rebalancing cases, Sharpe ratio, and out-of-sample utility. The optimal results are further grouped across different gammas ranging from 1 to 5 and call and put options separately. The portfolio optimization is conducted under the solvency constraint. The sample period spans from January 1996 to August 2023.

Table 6
Option Portfolio Performance with Solvency Constraints and Transaction Cost

	Stock	Call Option					Put Option				
		Moneyness (K/S)					Moneyness (K/S)				
		0.95	0.98	1	1.02	1.05	0.95	0.98	1	1.02	1.05
Panel A: Gamma=1											
w_o		0.078	0.014	0.010	0.009	0.004	0.004	0.002	0.011	0.048	0.081
$w_o\% > 0$		56.928	45.783	42.470	45.783	44.578	56.928	44.277	41.867	46.386	51.506
w_s	0.999	0.918	0.967	0.905	0.869	0.911	2.947	1.825	1.983	2.790	2.669
$w_s\% > 0$	100.000	99.398	99.398	97.590	95.482	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.138	0.053	0.064	0.091	0.118	0.143	0.063	0.067	0.098	0.121	0.156
OOSU	0.006	−0.001	0.002	0.004	0.007	0.008	−0.001	0.002	0.005	0.006	0.007
Panel B: Gamma=2											
w_o		0.015	−0.009	−0.005	0.000	0.000	−0.002	−0.006	−0.004	0.009	0.020
$w_o\% > 0$		46.687	36.145	35.241	39.157	39.759	44.880	34.639	34.940	39.458	46.687
w_s	0.991	0.978	0.992	0.944	0.904	0.925	1.519	0.919	0.978	1.384	1.394
$w_s\% > 0$	100.000	99.699	99.398	97.590	95.482	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.142	0.082	0.105	0.128	0.152	0.176	0.086	0.106	0.133	0.158	0.181
OOSU	0.005	0.001	0.003	0.004	0.005	0.006	0.001	0.003	0.004	0.005	0.006
Panel C: Gamma=3											
w_o		−0.007	−0.016	−0.010	−0.004	−0.001	−0.004	−0.008	−0.009	−0.003	0.001
$w_o\% > 0$		37.048	27.410	28.012	35.241	36.145	37.651	28.916	27.711	36.446	44.880
w_s	0.967	0.997	0.998	0.955	0.911	0.921	1.016	0.616	0.645	0.915	0.980
$w_s\% > 0$	100.000	99.699	99.398	97.590	95.482	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.145	0.102	0.126	0.147	0.167	0.183	0.105	0.127	0.151	0.172	0.186
OOSU	0.004	0.001	0.003	0.004	0.005	0.005	0.001	0.003	0.004	0.005	0.005
Panel D: Gamma=5											
w_o		−0.025	−0.023	−0.013	−0.006	−0.002	−0.006	−0.010	−0.013	−0.012	−0.012
$w_o\% > 0$		28.012	19.578	21.084	30.120	29.217	25.904	17.169	18.072	30.422	38.554
w_s	0.858	1.012	0.999	0.955	0.900	0.893	0.617	0.366	0.379	0.559	0.654
$w_s\% > 0$	100.000	99.699	99.398	97.590	95.482	95.181	100.000	100.000	100.000	100.000	100.000
Sharpe Ratio	0.129	0.127	0.143	0.150	0.169	0.178	0.129	0.141	0.154	0.174	0.179
OOSU	0.001	0.001	0.002	0.002	0.003	0.003	0.001	0.002	0.002	0.003	0.003

This table summarizes the performance of the optimal portfolio with options. As a comparison, we include the optimal portfolio with stock only. For each portfolio with options, we only include one option with the moneyness specified at the top of each column. The portfolio is rebalanced by the end of the third Friday of each month with the same moneyness over time. The reported statistics include the average portfolio weights for stocks and options, the proportion of positive weights over all rebalancing cases, Sharpe ratio, and out-of-sample utility. The optimal results are further grouped across different gammas ranging from 1 to 5 and call and put options separately. The portfolio optimization is conducted under the solvency constraint and considered transaction cost (i.e., buy at the ask price and sell at the bid price). The sample period spans from January 1996 to August 2023.

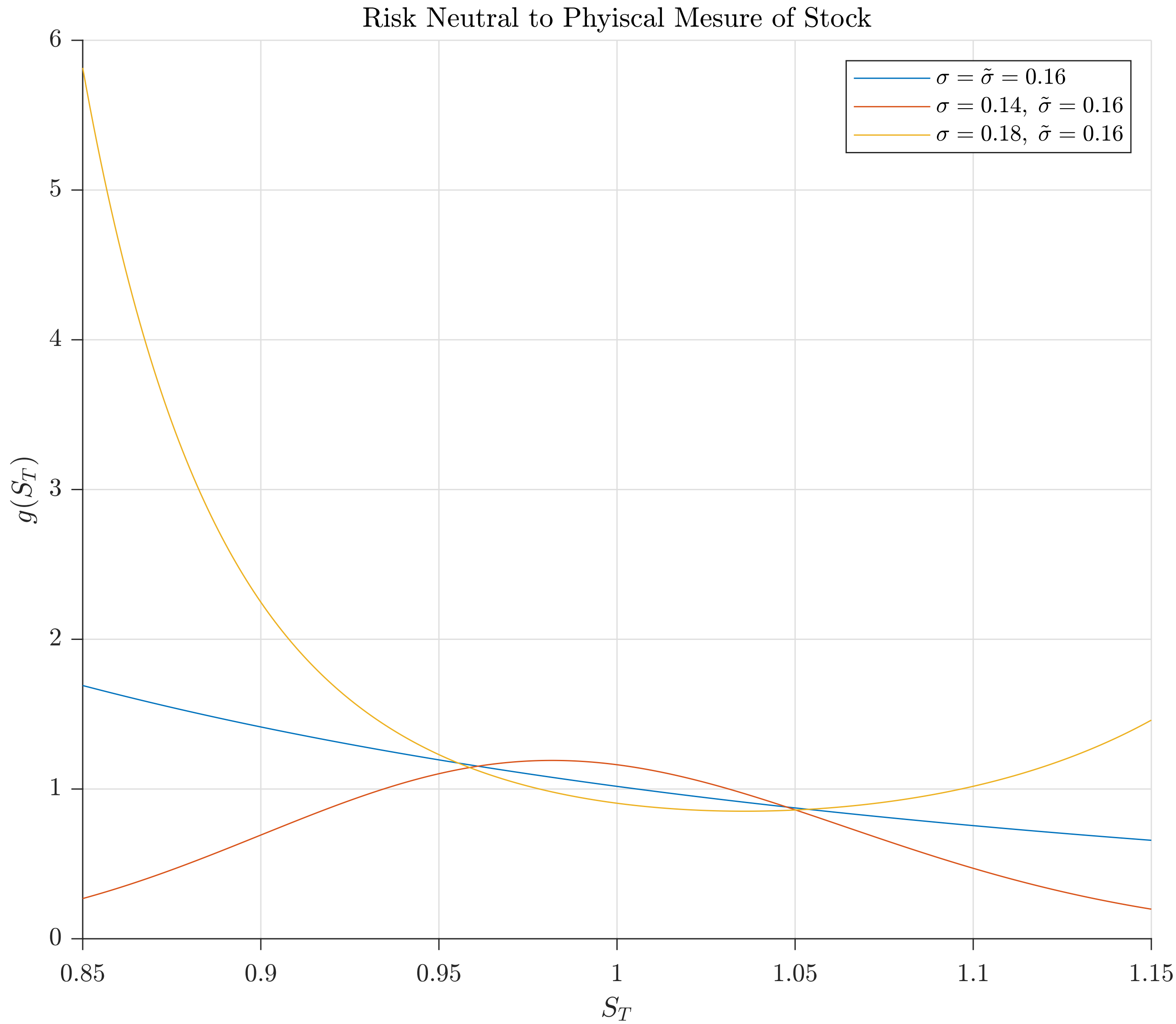

Figure 1
Plot of $g(S_T)$ for Different Values of σ
This figure plots $g(S_T)$ defined in equation (16) for three different cases. Blue line plots the baseline case when $\sigma = \tilde{\sigma} = 0.16$, red line plots the case when the risk-neutral volatility $\sigma = 0.14$ is lower than the physical volatility $\tilde{\sigma} = 0.16$, and the orange line plots the case when the risk-neutral volatility $\sigma = 0.18$ is greater than the physical volatility $\tilde{\sigma} = 0.16$. In all cases, we use $\mu = 0.12$ and $r = 0.04$.

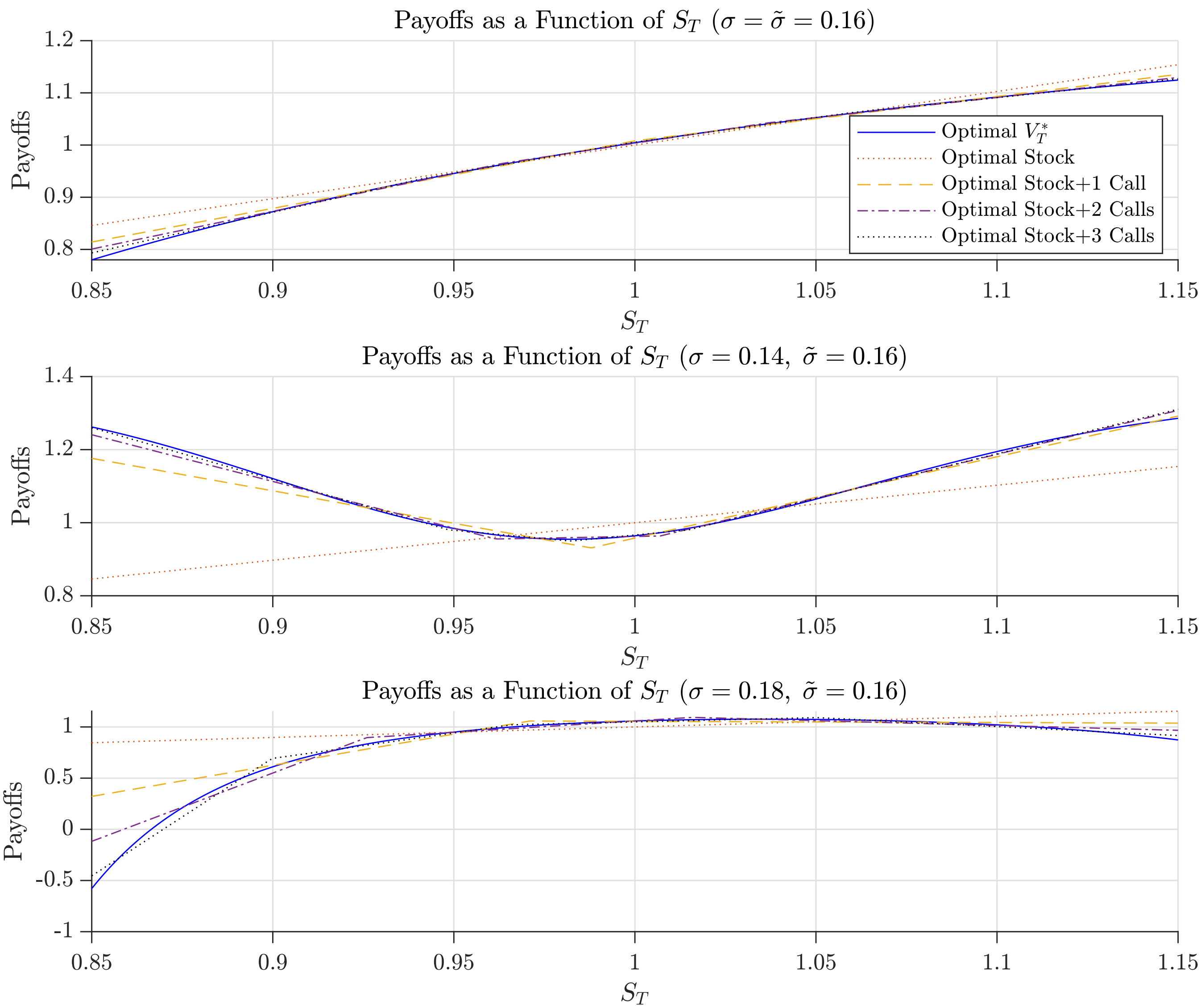

Figure 2
Payoffs of Optimal Option Portfolios
This figure plots the payoffs of optimal portfolios as a function of terminal stock price S_T. We assume a one-month holding period for portfolio and all options have expiry of one-month as well. We consider three cases in each panel. The top panel corresponds to $\sigma = \tilde{\sigma} = 0.16$, the middle panel corresponds to $\sigma = 0.14$ is less than $\tilde{\sigma} = 0.16$, and the bottom panel corresponds to $\sigma = 0.18$ is greater than $\tilde{\sigma} = 0.16$. The blue line plots the optimal benchmark payoff V_T^* that can be obtained assuming options of all range of strike prices are available. We then plot the corresponding payoffs of 4 optimal portfolios consisting of stock only, stock plus 1 call option, stock plus 2 call options, and stock plus 3 call options. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

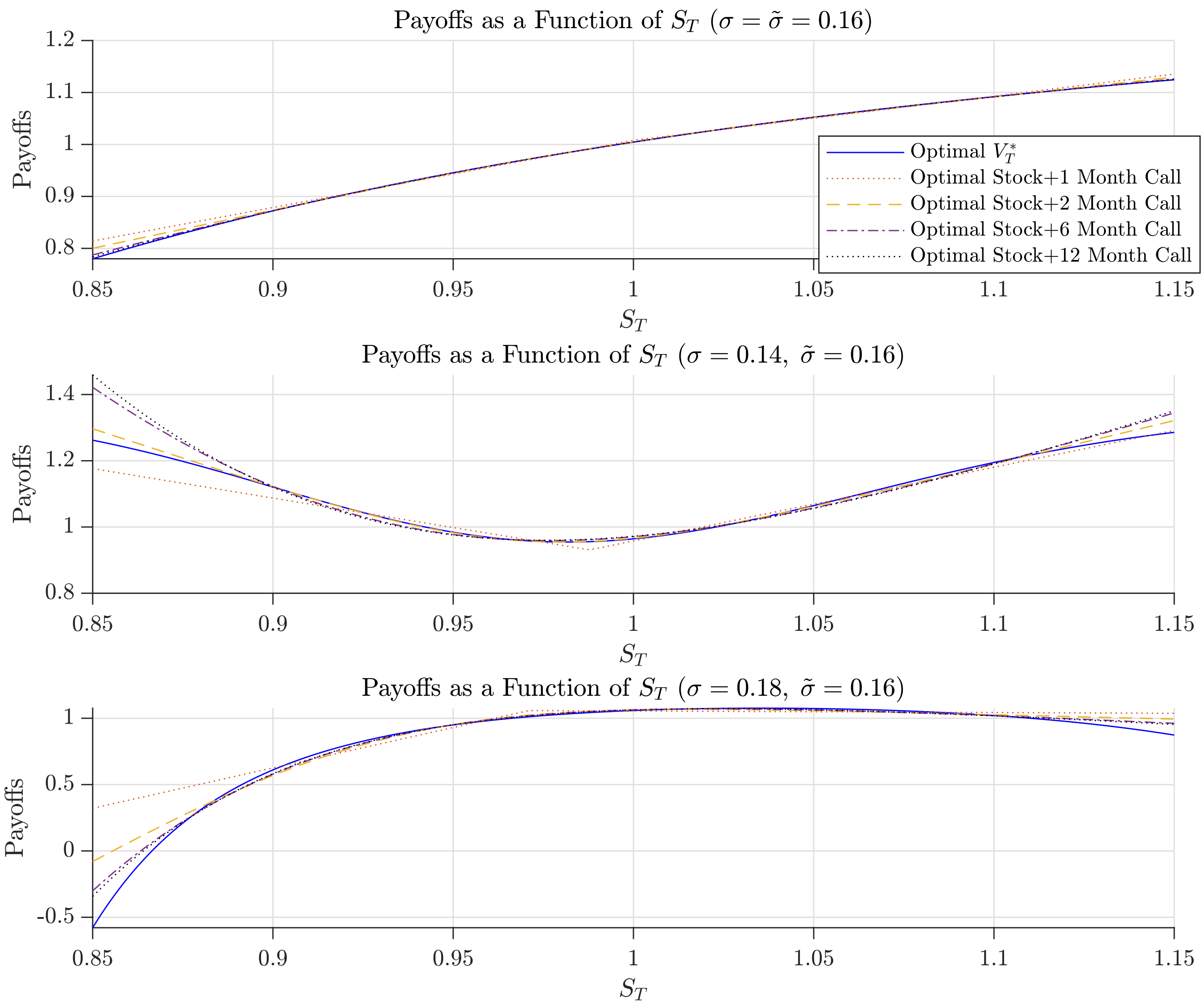

Figure 3
Payoffs of Optimal Option Portfolios using Options with Different Maturities
This figure plots the payoffs of optimal portfolios as a function of terminal stock price S_T. We assume a one-month holding period for portfolio. We consider three cases in each panel. The top panel corresponds to $\sigma = \tilde{\sigma} = 0.16$, the middle panel corresponds to $\sigma = 0.14$ is less than $\tilde{\sigma} = 0.16$, and the bottom panel corresponds to $\sigma = 0.18$ is greater than $\tilde{\sigma} = 0.16$. The blue line plots the optimal benchmark payoff V_T^* that can be obtained assuming options of all range of strike prices are available. We then plot the corresponding payoffs of 4 optimal portfolios consisting stock plus a call option maturing in 1, 2, 6, and 12 months. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

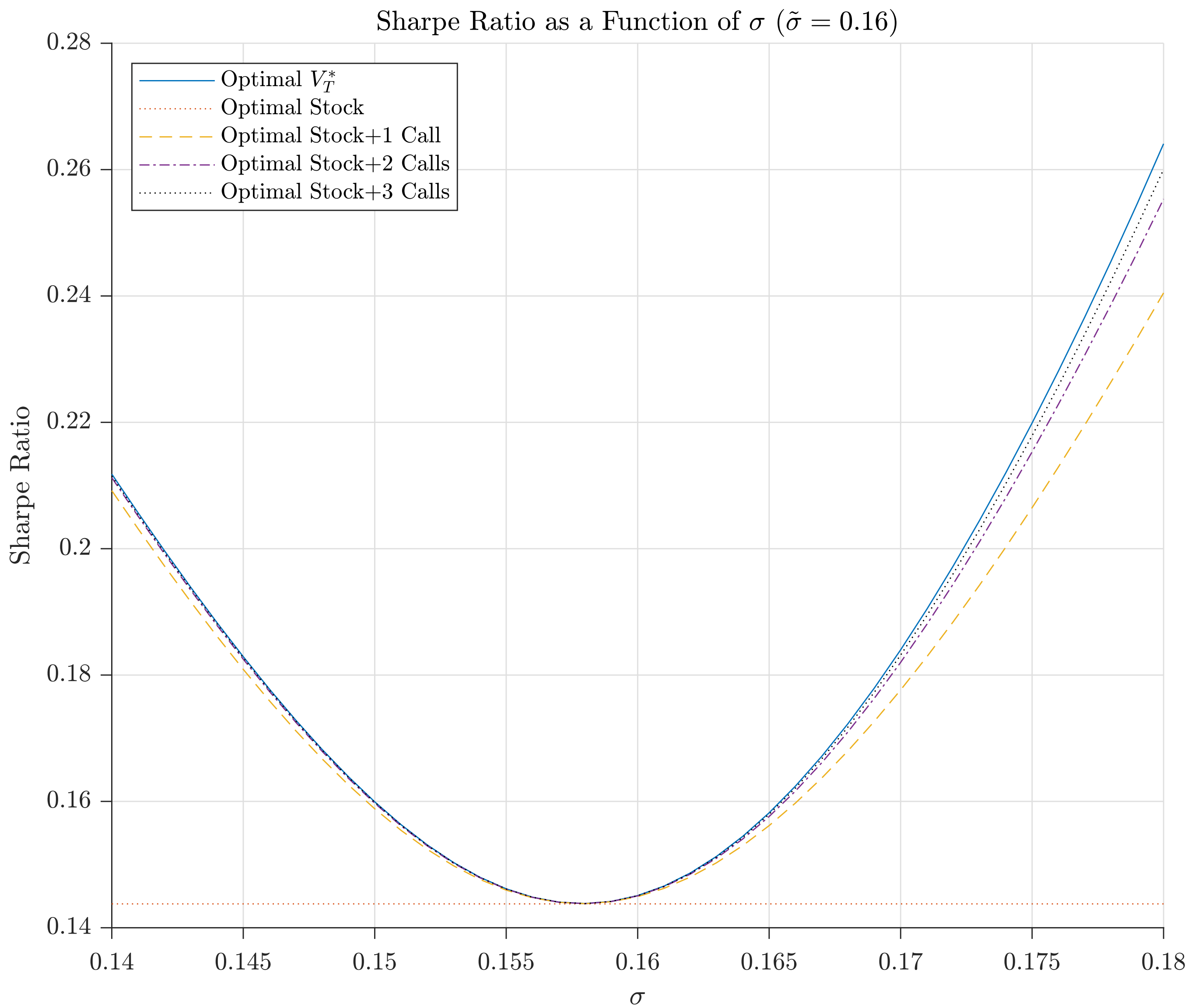

Figure 4
Sharpe Ratio of Optimal Option Portfolios

This figure plots the Sharpe ratios of optimal portfolios as a function of risk-neutral volatility σ. We assume a one-month holding period for portfolio and all options have expiry of one-month as well. The blue line plots the Sharpe ratio of the optimal benchmark portfolio V_T^* that can be obtained assuming options of all range of strike prices are available. We then plot the corresponding Sharpe ratios of 4 optimal portfolios consisting of stock only, stock plus 1 call option, stock plus 2 call options, and stock plus 3 call options. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $\tilde{\sigma} = 0.16$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

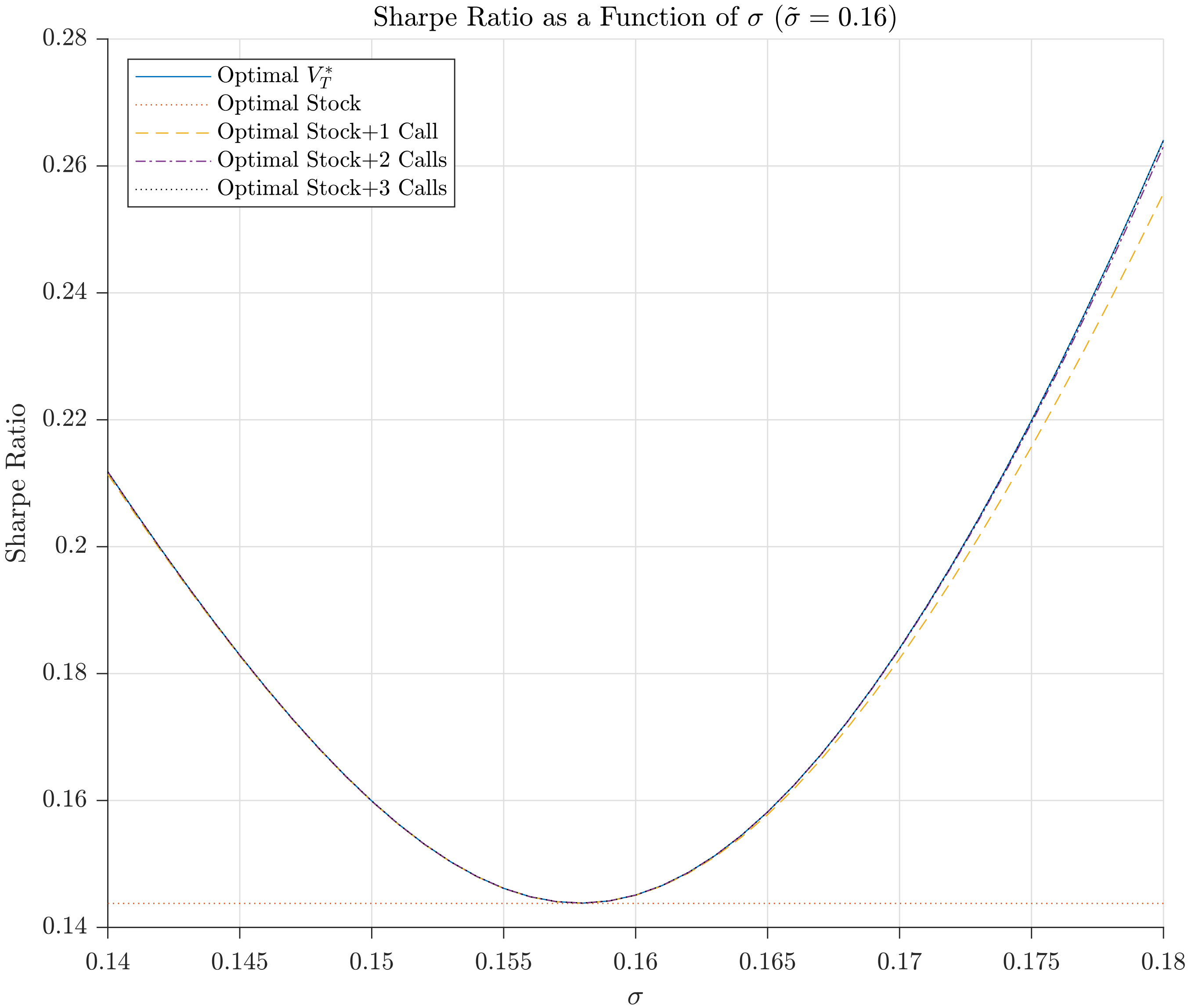

Figure 5
Sharpe Ratio of Optimal Option Portfolios with Options Maturing in 2 Months
This figure plots the Sharpe ratios of optimal portfolios as a function of risk-neutral volatility σ. We assume a one-month holding period for portfolio and all options have expiry of two-months. The blue line plots the Sharpe ratio of the optimal benchmark portfolio V_T^* that can be obtained assuming options of all range of strike prices are available. We then plot the corresponding Sharpe ratios of 4 optimal portfolios consisting of stock only, stock plus 1 call option, stock plus 2 call options, and stock plus 3 call options. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $\tilde{\sigma} = 0.16$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

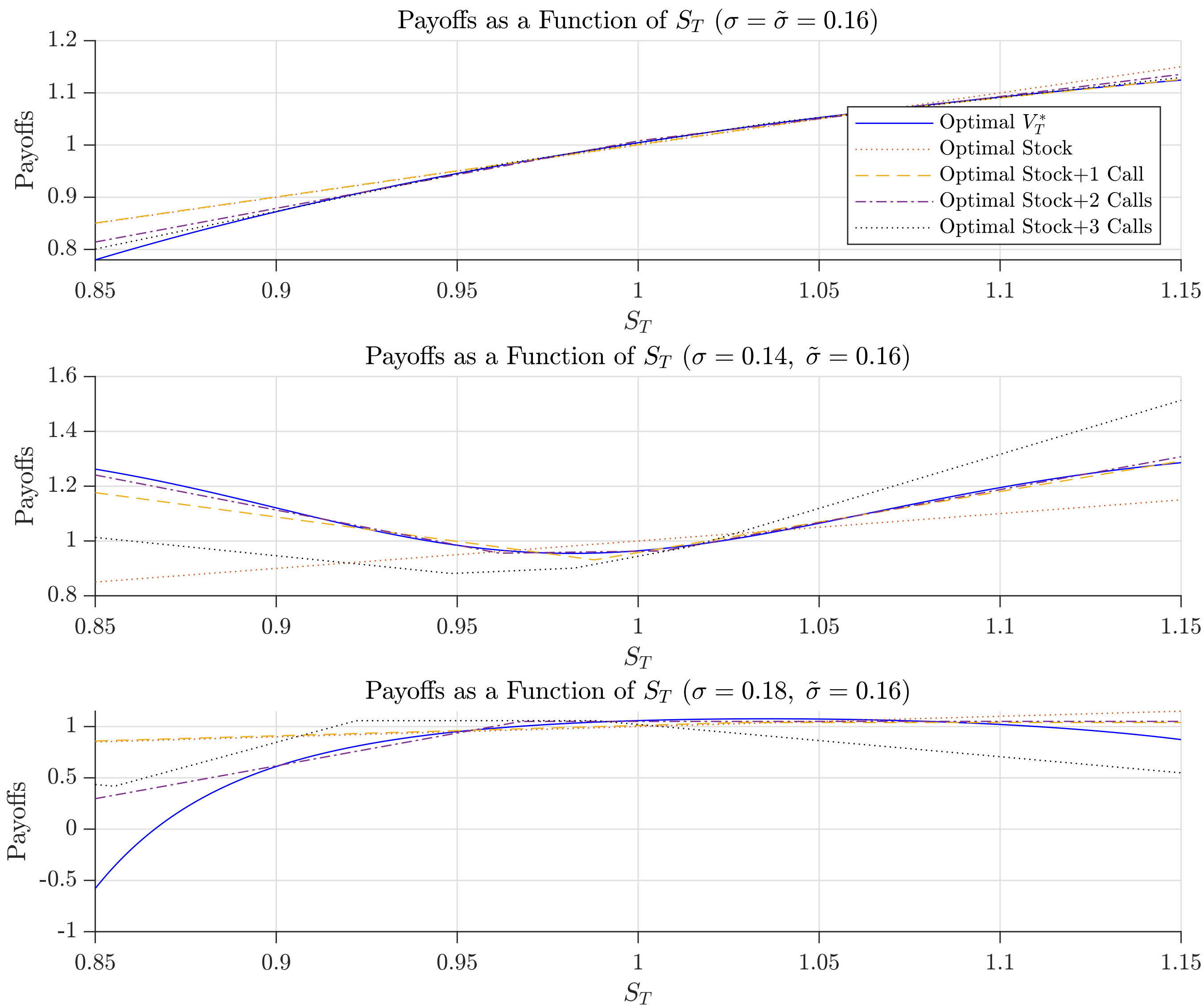

Figure 6
Payoffs of Optimal Option Portfolios with Solvency Constraint
This figure plots the payoffs of optimal portfolios as a function of terminal stock price S_T with the solvency constraint requiring that $V_T \geq 0$ for all values of S_T. We assume a one-month holding period for portfolio and all options have expiry of one-month as well. We consider three cases in each panel. The top panel corresponds to $\sigma = \tilde{\sigma} = 0.16$, the middle panel corresponds to $\sigma = 0.14$ is less than $\tilde{\sigma} = 0.16$, and the bottom panel corresponds to $\sigma = 0.18$ is greater than $\tilde{\sigma} = 0.16$. The blue line plots the optimal benchmark payoff V_T^* that can be obtained assuming options of all range of strike prices are available. We then plot the corresponding payoffs of 4 optimal portfolios consisting of stock only, stock plus 1 call option, stock plus 2 call options, and stock plus 3 call options. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

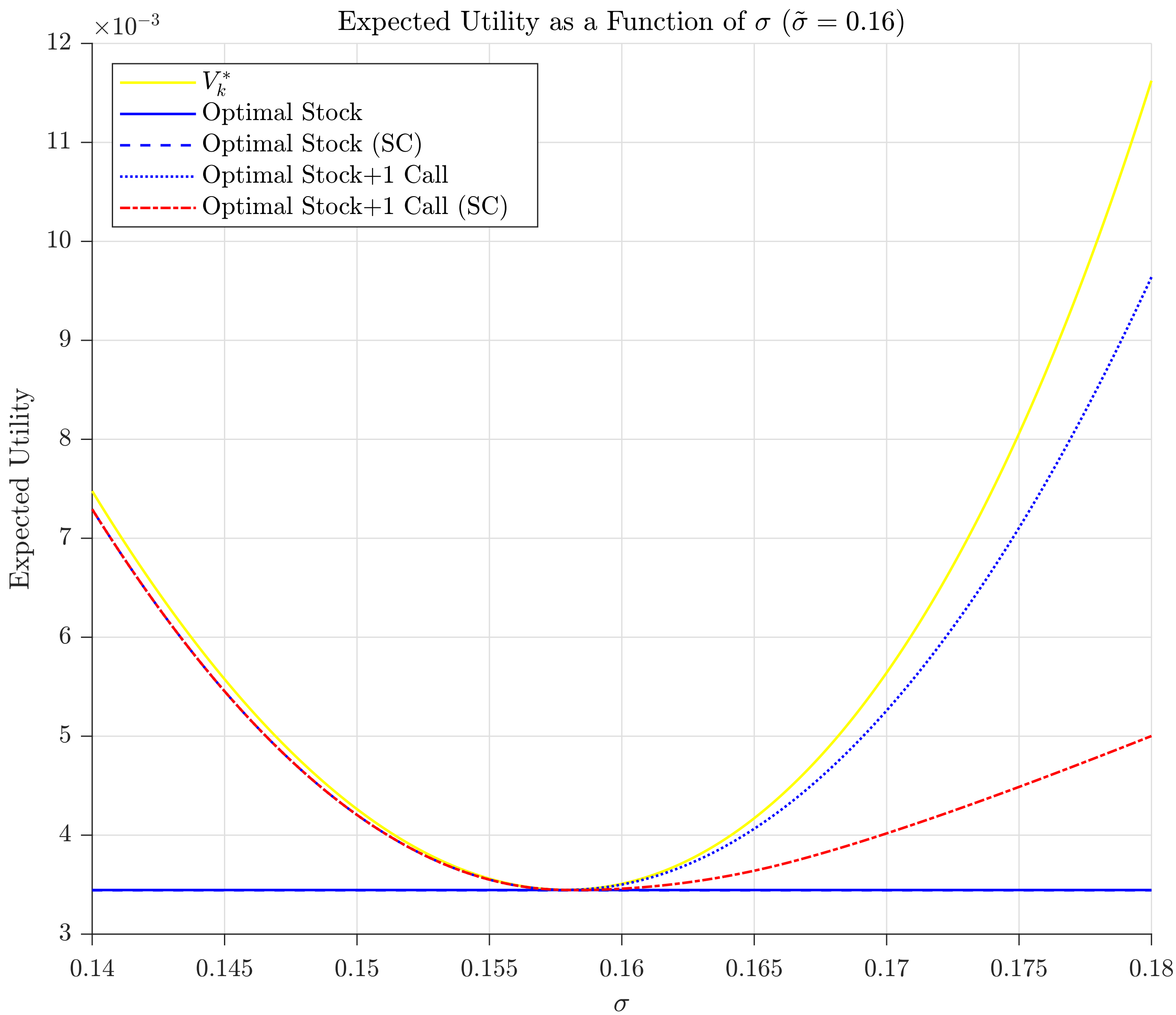

Figure 7
Expected Utility of Optimal Option Portfolios with Solvency Constraint ($\gamma = 3$)
This figure plots the expected utility of optimal portfolios as a function of risk-neutral volatility σ with the solvency constraint requiring that $V_T \geq 0$ for all values of S_T. We assume a one-month holding period for portfolio and all options have expiry of one-month as well. For portfolio with stock only and stock and a call option, we plot both unconstrained case and constrained case. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

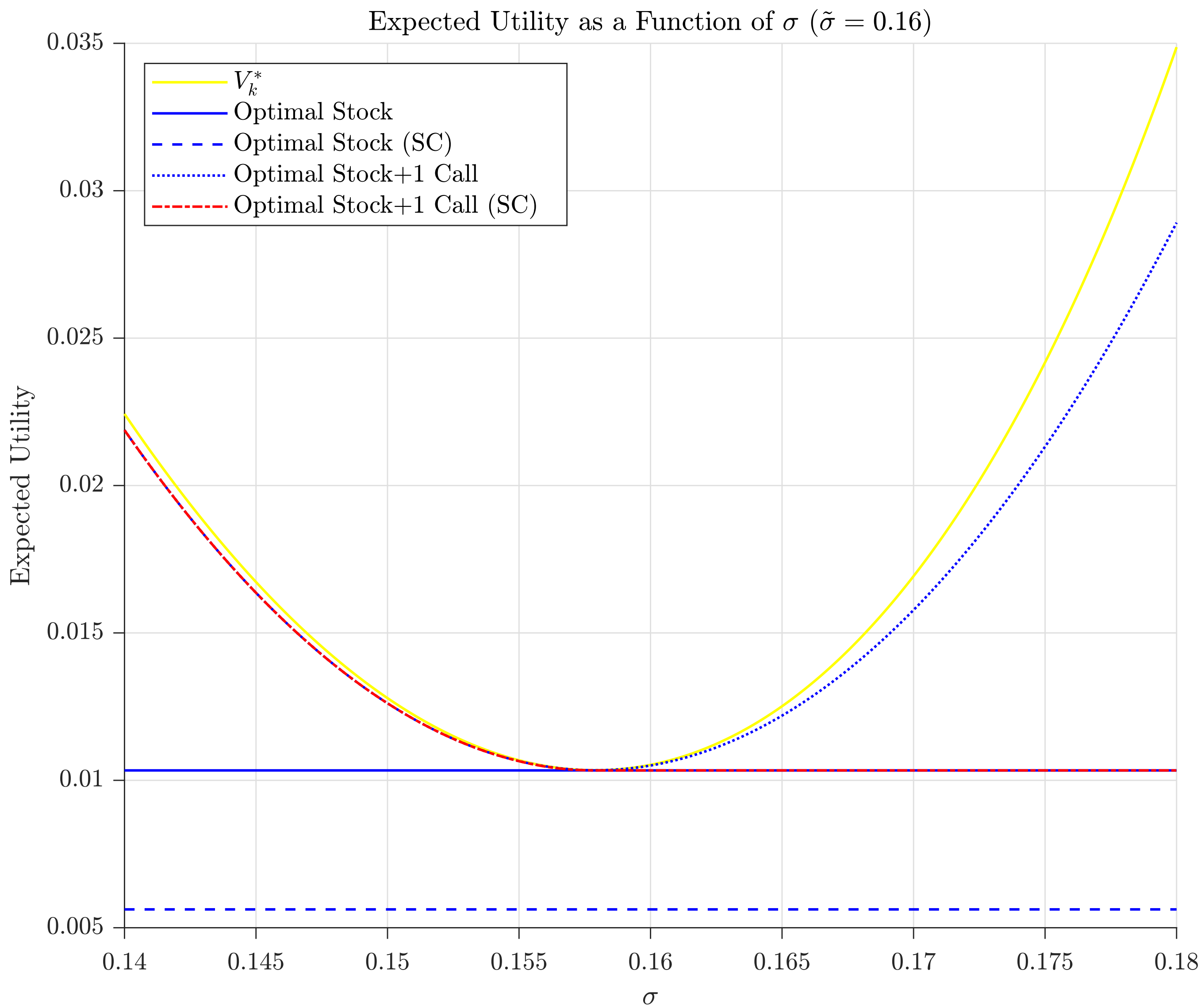

Figure 8
Expected Utility of Optimal Option Portfolios with Solvency Constraint ($\gamma = 1$)
This figure plots the expected utility of optimal portfolios as a function of risk-neutral volatility σ with the solvency constraint requiring that $V_T \geq 0$ for all values of S_T. We assume a one-month holding period for portfolio and all options have expiry of one-month as well. For portfolio with stock only and stock and a call option, we plot both unconstrained case and constrained case. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 1$.

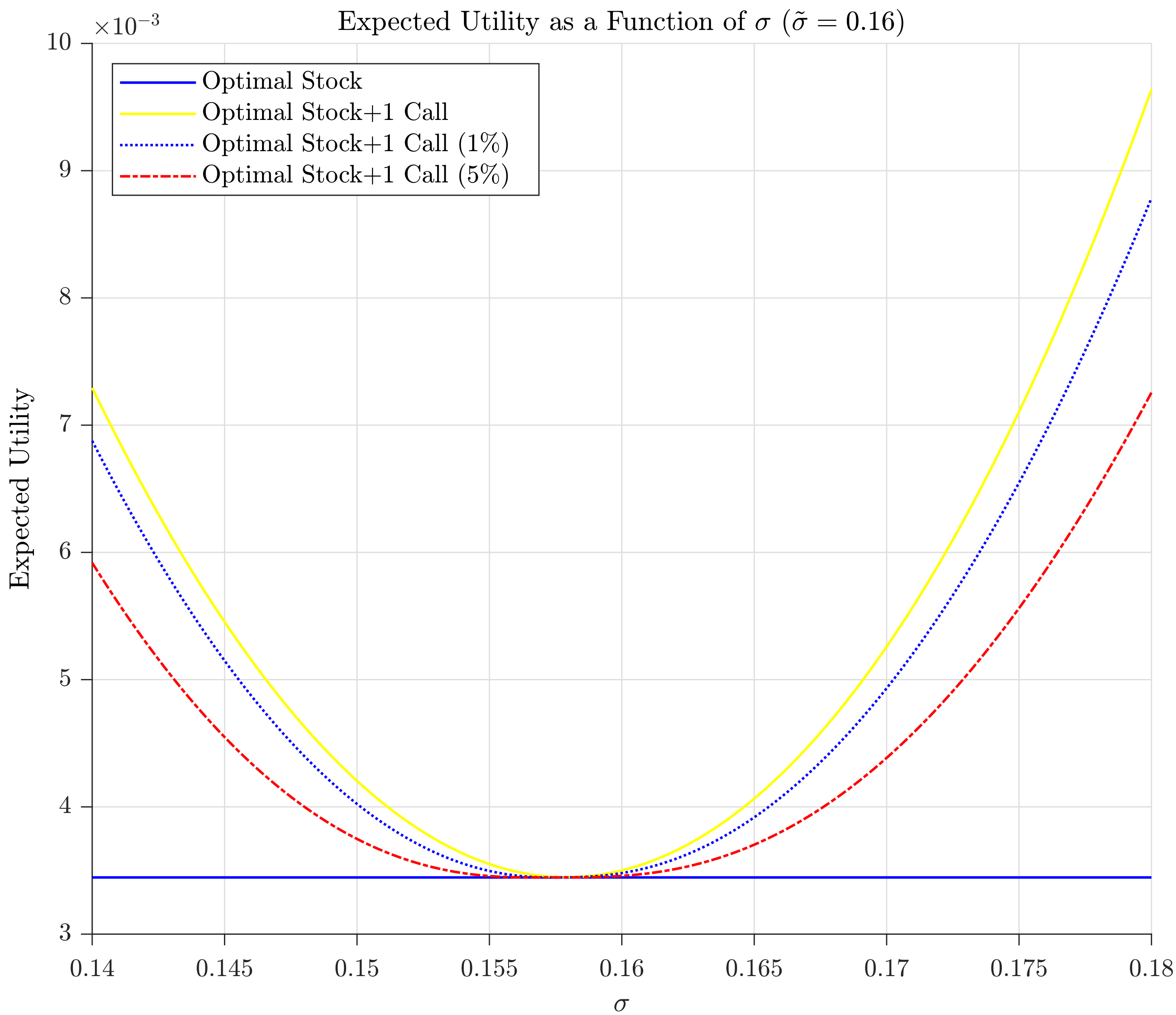

Figure 9
Expected Utility of Optimal Option Portfolios with Bid-Ask Spreads
This figure plots the expected utility of optimal portfolios as a function of risk-neutral volatility σ with bid-ask spreads. We assume a one-month holding period for portfolio and all options have expiry of one-month as well. For portfolio with stock only and stock and a call option, we plot cases corresponding to stock only, 0%, 1%, and 5% bid-ask spreads of the option price. Strike prices of call options are optimally chosen to maximize the mean-variance utility of the portfolio. We use the following parameters to optimize the portfolio: $\mu = 0.12$, $r = 0.04$, $S_0 = 1$, $\delta = 0$, and $\gamma = 3$.

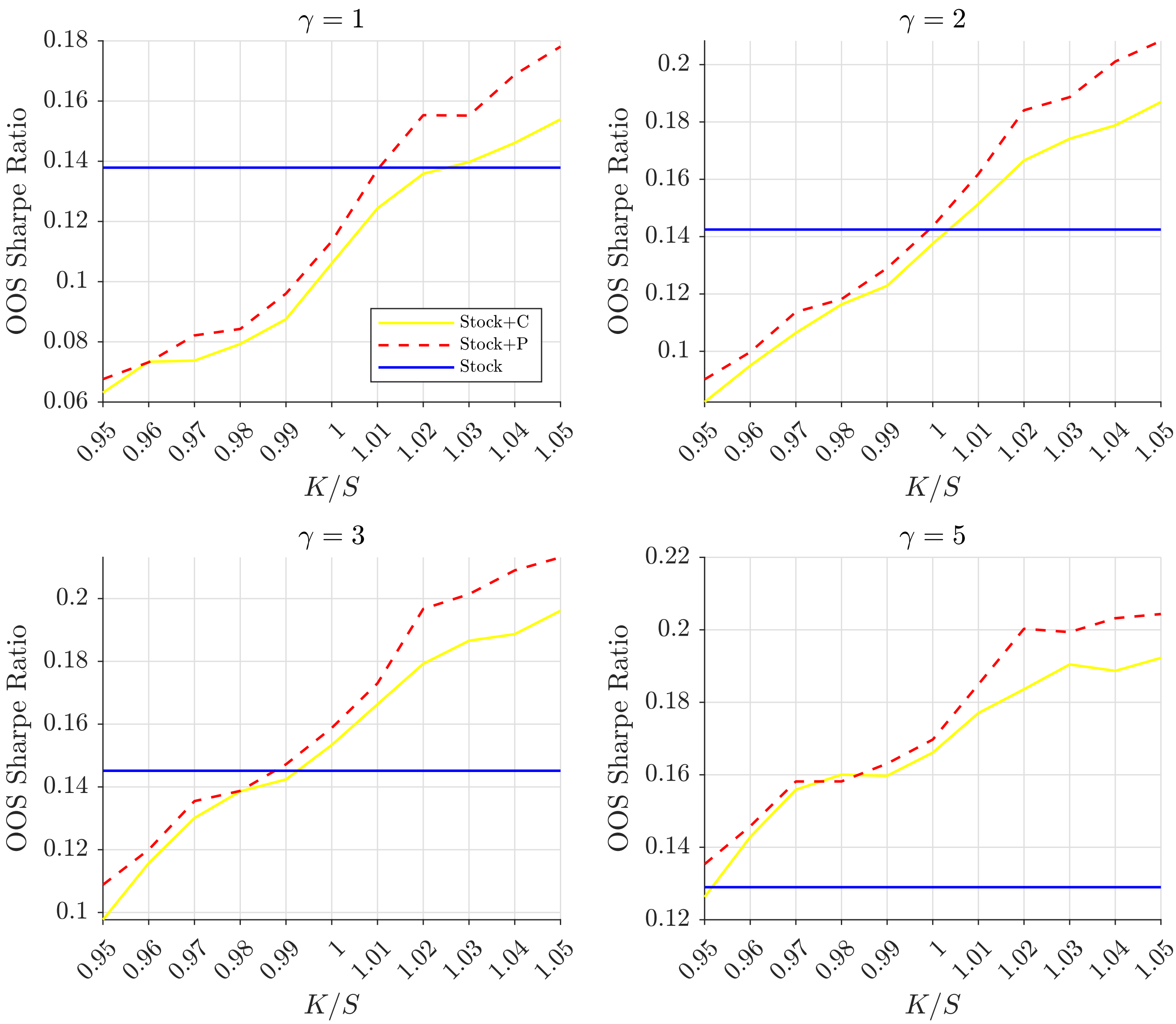

Figure 10
Out-of-Sample Sharpe Ratios of Optimal Portfolios with Solvency Constraint
This figure plots the out-of-sample (OOS) Sharpe ratio of optimal portfolio consisting of S&P500 index and a call or put option as a function of moneyness of the option. Each panel corresponds to the different value of investors' risk aversion parameter γ ranging from 1 to 5. Portfolio is constructed at the third Friday of each month using options with one-month to maturity and held until the maturity. A solvency constraint is imposed to ensure that the portfolio value remains strictly non-negative at maturity. The sample period is from January 1996 to August 2023.

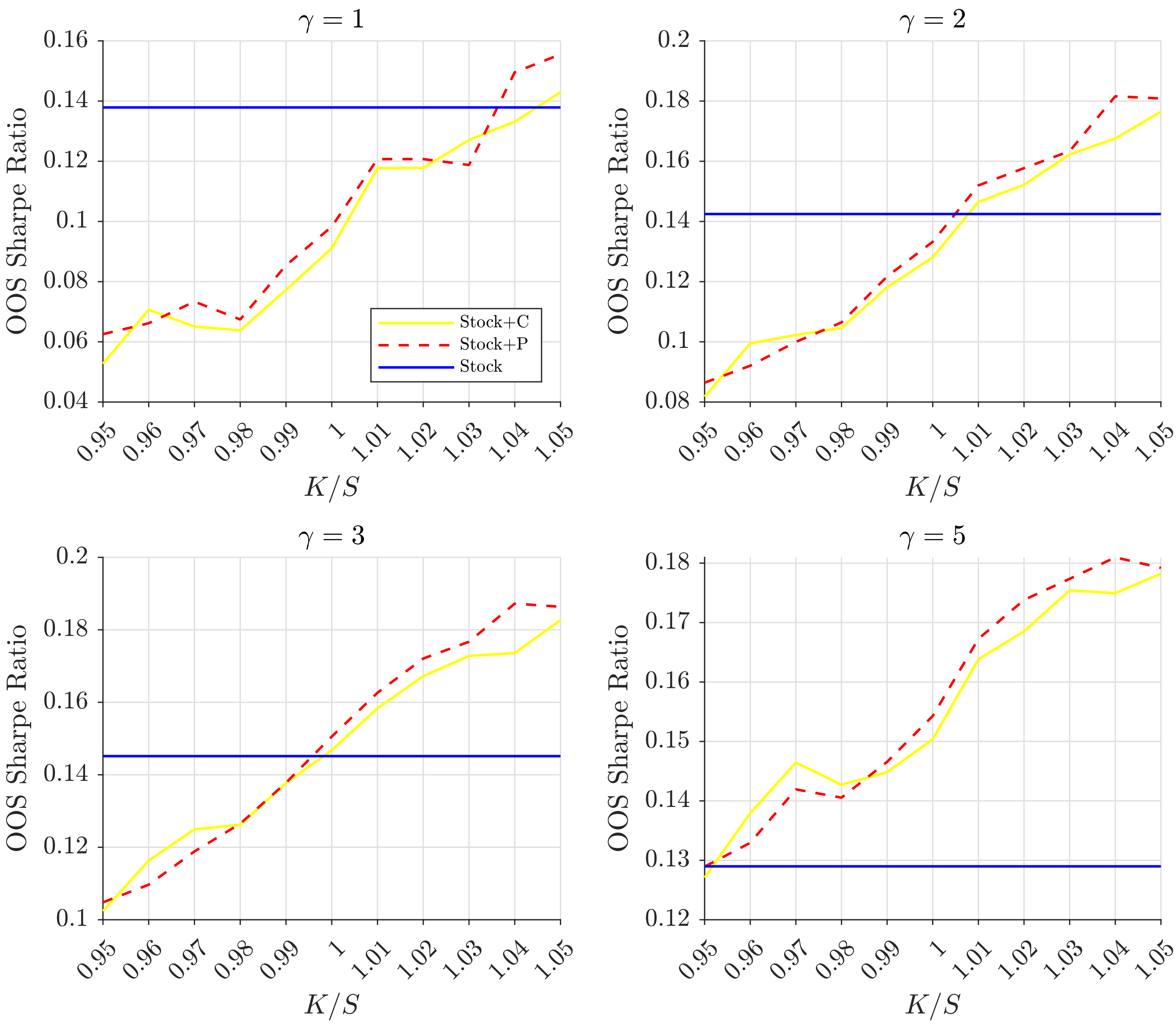

Figure 11

Out-of-Sample Sharpe Ratios of Optimal Portfolios with Solvency Constraint and Transaction Costs

This figure plots the out-of-sample (OOS) Sharpe ratio of optimal portfolio consisting of S&P500 index and a call or put option as a function of moneyness of the option. Each panel corresponds to the different value of investors' risk aversion parameter γ ranging from 1 to 5. Portfolio is constructed at the third Friday of each month using options with one-month to maturity and held until the maturity. A solvency constraint is imposed to ensure that the portfolio value remains strictly non-negative at maturity. Moreover, full transaction cost is assumed so that options are assumed to be bought at the best ask price and sold at the best bid price. The sample period is from January 1996 to August 2023.

Online Appendix for
"Optimal Portfolio with Options"

January 27, 2026

This document supplements the paper "Optimal Portfolio with Options". It provides additional results and robustness analyses which are not displayed in the published text.

OA.1 Moments of Future Value of a European Option (Holding Period Differs from Maturity)

Suppose today is $t = 0$ and we assume the stock price at time k is log-normally distributed, which implies

$$\log(S_k) \sim N\left(\log(S_0) + k\left(\mu - \delta - \frac{\sigma_k^2}{2}\right), k\sigma_k^2\right), \tag{1}$$

where μ is the drift of the geometric Brownian motion, δ is the continuously compounded dividend yield, and σ_k is the (annualized) variance of the k-period continuously compounded return of the stock. Note that in general, σ_k^2 differs across k when returns of the stock are autocorrelated.

We are interested in computing the moments of the value of a European call option at time k, where the option has maturity at τ, with $k < \tau$. We assume the value of a European call option at time k is given by

$$C_k(K) = S_k e^{-\delta(\tau-k)}\Phi(d_{1,k}) - Ke^{-r(\tau-k)}\Phi(d_{2,k}), \tag{2}$$

where

$$d_{1,k} = \frac{\log(S_k/K) + (\tau - k)(r - \delta) + \frac{(\tau-k)\sigma^2}{2}}{\sqrt{\tau - k}\sigma}, \tag{3}$$

$$d_{2,k} = d_{1,k} - \sqrt{\tau - k}\sigma. \tag{4}$$

Here, we assume the Black-Scholes formula holds for the call price and σ for the risk-neutral process is different from σ_k or σ_τ for the physical process. However, such an assumption is not needed when $k = \tau$ because $C_\tau(K) = \max[0, S_\tau - K]$ and it does not depend on the choice of option pricing model.

We are interested in computing $E[C_k(K)^m]$. Without loss of generality, we can assume $S_0 = 1$.

In addition, we are interested in computing $E[P_k(K)^m]$, $E[S_k^{m_1} C_k(K)^{m_2}]$, $E[S_k^{m_1} P_k(K)^{m_2}]$, $E[C_k(K_1)^{m_1} C_k(K_2)^{m_2}]$, $E[P_k(K_1)^{m_1} P_k(K_2)^{m_2}]$, and $E[C_k(K_1)^{m_1} P_k(K_2)^{m_2}]$.

Assuming $S_0 = 1$, we have $\log(S_k) \sim N\left(k\left(\mu - \delta - \frac{\sigma_k^2}{2}\right), k\sigma_k^2\right)$. From this, we have

$$E_t[C_k(K)^m] = \int_0^\infty (S_k e^{-\delta(\tau-k)}\Phi(d_{1,k}) - Ke^{-r(\tau-k)}\Phi(d_{2,k}))^m f(S_k) dS_k$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} (e^x e^{-\delta(\tau-k)}\Phi(d_{1,k}) - Ke^{-r(\tau-k)}\Phi(d_{2,k}))^m \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= \sum_{i=0}^{m} \binom{m}{i} e^{-i\delta(\tau-k)}(-K)^{m-i} e^{-r(\tau-k)(m-i)} \\
&\quad \times \int_{-\infty}^{\infty} e^{ix}\Phi(d_{1,k})^i \Phi(d_{2,k})^{m-i} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= \sum_{i=0}^{m} \binom{m}{i} e^{-i\delta(\tau-k)}(-K)^{m-i} e^{-r(\tau-k)(m-i)} e^{(i(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{i^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^{\infty} \Phi(d_{1,k})^i \Phi(d_{2,k})^{m-i} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta+(i-\frac{1}{2})\sigma_k^2)k)^2}{2k\sigma_k^2}} dx \\
&= \sum_{i=0}^{m} \binom{m}{i} e^{-i\delta(\tau-k)}(-K)^{m-i} e^{-r(\tau-k)(m-i)} e^{(i(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{i^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^{\infty} \Phi(A_1(i)+Bz)^i \Phi(A_2(i)+Bz)^{m-i} \phi(z) dz,
\end{aligned}
$$

where we have used two change of variables: $x = \log(S_k)$ and $z = \frac{x-(\mu-\delta+(i-\frac{1}{2})\sigma_k^2)k}{\sqrt{k}\sigma_k}$. Expressions for As and Bs are given below:

$$
\begin{aligned}
A_1(i) &= \frac{(\mu-\delta+(i-\frac{1}{2})\sigma_k^2)k - \log(K) + (\tau-k)(r-\delta) + \frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma}, \\
A_2(i) &= A_1(i) - \sqrt{\tau-k}\sigma, \\
B &= \frac{\sqrt{k}\sigma_k}{\sqrt{\tau-k}\sigma}.
\end{aligned}
$$

To compute 2^{nd} moment, we can use Owen (1980)'s result:

$$
\int_{-\infty}^{\infty} \Phi(A+Bz)\Phi(C+Dz)\phi(z)dz = \mathrm{BvN}\left(\frac{A}{\sqrt{1+B^2}}, \frac{C}{\sqrt{1+D^2}}, \rho = \frac{BD}{\sqrt{1+B^2}\sqrt{1+D^2}}\right).
$$

To compute 3^{rd} moment, we use the following result of Owen (1980):

$$
\begin{aligned}
\int_{-\infty}^{\infty} \Phi(A+Bz)\Phi(C+Dz)\Phi(E+Fz)\phi(z)dz = \mathrm{TvN}\Bigg(&\frac{A}{\sqrt{1+B^2}}, \frac{C}{\sqrt{1+D^2}}, \frac{E}{\sqrt{1+F^2}}; \\
\rho_{12} = \frac{BD}{\sqrt{1+B^2}\sqrt{1+D^2}}, \rho_{13} = \frac{BF}{\sqrt{1+B^2}\sqrt{1+F^2}}, \rho_{23} &= \frac{DF}{\sqrt{1+D^2}\sqrt{1+F^2}}\Bigg).
\end{aligned}
$$

Similarly, we can compute other moments involving stock, call, and put options. Moments of put options are give below

$$
\begin{aligned}
E_t[P_k(K)^m] &= \int_0^{\infty} (Ke^{-r(\tau-k)}\Phi(-d_{2,k}) - S_k e^{-\delta(\tau-k)}\Phi(-d_{1,k}))^m f(S_k) dS_k \\
&= \int_{-\infty}^{\infty} (Ke^{-r(\tau-k)}\Phi(-d_{2,k}) - e^x e^{-\delta(\tau-k)}\Phi(-d_{1,k}))^m \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= \sum_{i=0}^{m} \binom{m}{i} K^i e^{-r(\tau-k)i}(-1)^{m-i} e^{-\delta(\tau-k)(m-i)} \\
&\quad \times \int_{-\infty}^{\infty} e^{(m-i)x}\Phi(-d_{2,k})^i \Phi(-d_{1,k})^{m-i} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= \sum_{i=0}^{m} \binom{m}{i} K^i e^{-r(\tau-k)i}(-1)^{m-i} e^{-\delta(\tau-k)(m-i)} e^{((m-i)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m-i)^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^{\infty} \Phi(-d_{2,k})^i \Phi(-d_{1,k})^{m-i} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta+((m-i)-\frac{1}{2})\sigma_k^2)k)^2}{2k\sigma_k^2}} dx \\
&= \sum_{i=0}^{m} \binom{m}{i} K^i e^{-r(\tau-k)i}(-1)^{m-i} e^{-\delta(\tau-k)(m-i)} e^{((m-i)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m-i)^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^{\infty} \Phi(A_1(i)+Bz)^i \Phi(A_2(i)+Bz)^{m-i} \phi(z) dz,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(i) &= -\frac{(\mu-\delta+(m-i-\frac{1}{2})\sigma_k^2)k - \log(K) + (\tau-k)(r-\delta-\sigma^2) + \frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma}, \\
A_2(i) &= A_1(i) - \sqrt{\tau-k}\sigma, \\
B &= -\frac{\sqrt{k}\sigma_k}{\sqrt{\tau-k}\sigma}.
\end{aligned}
$$

Next, we would like to compute the cross-moment of stock and call option $E[\tilde{S_k}^{m_1} C_k(K)^{m_2}]$. Here, we use $\tilde{S_k}$ to denote the stock price at time k including the dividends paid out from t to $t+k$. Thus, we can write $\tilde{S_k} = S_k e^{\delta k}$. In other words, we have $\log(\tilde{S_k}) \sim N\left(k\left(\mu - \frac{\sigma_k^2}{2}\right), k\sigma_k^2\right)$. Then, the cross-moment becomes

$$
\begin{aligned}
& E_t[\tilde{S_k}^{m_1} C_k(K)^{m_2}] \\
&= \int_0^{\infty} \tilde{S_k}^{m_1} (S_k e^{-\delta(\tau-k)}\Phi(d_{1,k}) - Ke^{-r(\tau-k)}\Phi(d_{2,k}))^{m_2} f(S_k) dS_k
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty S_k^{m_1} e^{\delta k m_1} (S_k e^{-\delta(\tau-k)}\Phi(d_{1,k}) - Ke^{-r(\tau-k)}\Phi(d_{2,k}))^{m_2} f(S_k) dS_k \\
&= \int_{-\infty}^\infty e^{m_1 x} e^{\delta k m_1} (e^x e^{-\delta(\tau-k)}\Phi(d_{1,k}) - Ke^{-r(\tau-k)}\Phi(d_{2,k}))^{m_2} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= e^{\delta k m_1} \sum_{i=0}^{m_2} \binom{m_2}{i} e^{-i\delta(\tau-k)} (-K)^{m_2-i} e^{-r(\tau-k)(m_2-i)} \\
&\quad \times \int_{-\infty}^\infty e^{(m_1+i)x} \Phi(d_{1,k})^i \Phi(d_{2,k})^{m_2-i} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= e^{\delta k m_1} \sum_{i=0}^{m_2} \binom{m_2}{i} e^{-i\delta(\tau-k)} (-K)^{m_2-i} e^{-r(\tau-k)(m_2-i)} e^{((m_1+i)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m_1+i)^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^\infty \Phi(d_{1,k})^i \Phi(d_{2,k})^{m_2-i} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta+((m_1+i)-\frac{1}{2})\sigma_k^2)k)^2}{2k\sigma_k^2}} dx \\
&= e^{\delta k m_1} \sum_{i=0}^{m_2} \binom{m_2}{i} e^{-i\delta(\tau-k)} (-K)^{m_2-i} e^{-r(\tau-k)(m_2-i)} e^{((m_1+i)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m_1+i)^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^\infty \Phi(A_1(i)+Bz)^i \Phi(A_2(i)+Bz)^{m_2-i} \phi(z) dz,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(i) &= \frac{(\mu-\delta+(m_1+i-\frac{1}{2})\sigma_k^2)k - \log(K) + (\tau-k)(r-\delta) + \frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma}, \\
A_2(i) &= A_1(i) - \sqrt{\tau-k}\sigma, \\
B &= \frac{\sqrt{k}\sigma_k}{\sqrt{\tau-k}\sigma}.
\end{aligned}
$$

Next, we would like to compute the cross-moment of stock and put option $E[\tilde{S}_k^{m_1} P_k(K)^{m_2}]$. The cross-moment becomes

$$
\begin{aligned}
&E_t[\tilde{S}_k^{m_1} P_k(K)^{m_2}] \\
&= \int_0^\infty \tilde{S}_k^{m_1} (Ke^{-r(\tau-k)}\Phi(-d_{2,k}) - S_k e^{-\delta(\tau-k)}\Phi(-d_{1,k}))^{m_2} f(S_k) dS_k \\
&= \int_0^\infty S_k^{m_1} e^{\delta k m_1} (Ke^{-r(\tau-k)}\Phi(-d_{2,k}) - S_k e^{-\delta(\tau-k)}\Phi(-d_{1,k}))^{m_2} f(S_k) dS_k \\
&= \int_{-\infty}^\infty e^{m_1 x} e^{\delta k m_1} (Ke^{-r(\tau-k)}\Phi(-d_{2,k}) - e^x e^{-\delta(\tau-k)}\Phi(-d_{1,k}))^{m_2} \frac{1}{\sqrt{2\pi k\sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx
\end{aligned}
$$

$$
\begin{aligned}
&= e^{\delta k m_1} \sum_{i=0}^{m_2} \binom{m_2}{i} K^i e^{-r(\tau-k)i} (-1)^{m_2-i} e^{-\delta(\tau-k)(m_2-i)} \\
&\quad \times \int_{-\infty}^{\infty} e^{(m_1+m_2-i)x} \Phi(-d_{2,k})^i \Phi(-d_{1,k})^{m-i} \frac{1}{\sqrt{2\pi k \sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= e^{\delta k m_1} \sum_{i=0}^{m_2} \binom{m_2}{i} K^i e^{-r(\tau-k)i} (-1)^{m_2-i} e^{-\delta(\tau-k)(m_2-i)} e^{((m_1+m_2-i)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m_1+m_2-i)^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^{\infty} \Phi(-d_{2,k})^i \Phi(-d_{1,k})^{m-i} \frac{1}{\sqrt{2\pi k \sigma_k^2}} e^{-\frac{(x-(\mu-\delta+((m_1+m_2-i)-\frac{1}{2})\sigma_k^2)k)^2}{2k\sigma_k^2}} dx \\
&= e^{\delta k m_1} \sum_{i=0}^{m_2} \binom{m_2}{i} K^i e^{-r(\tau-k)i} (-1)^{m_2-i} e^{-\delta(\tau-k)(m_2-i)} e^{((m_1+m_2-i)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m_1+m_2-i)^2\sigma_k^2}{2})k} \\
&\quad \times \int_{-\infty}^{\infty} \Phi(A_1(i)+Bz)^i \Phi(A_2(i)+Bz)^{m_2-i} \phi(z) dz,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(i) &= -\frac{(\mu-\delta+(m_1+m_2-i-\frac{1}{2})\sigma_k^2)k - \log(K) + (\tau-k)(r-\delta-\sigma^2) + \frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma}, \\
A_2(i) &= A_1(i) - \sqrt{\tau-k}\sigma, \\
B &= -\frac{\sqrt{k}\sigma_k}{\sqrt{\tau-k}\sigma}.
\end{aligned}
$$

Next, we compute the cross-moment of two call options with different strike prices K_1 and K_2

$$
\begin{aligned}
&E_t[C_k(K_1)^{m_1} C_k(K_2)^{m_2}] \\
&= \int_0^{\infty} (S_k e^{-\delta(\tau-k)} \Phi(d_{1,k;K_1}) - K_1 e^{-r(\tau-k)} \Phi(d_{2,k;K_1}))^{m_1} \\
&\qquad (S_k e^{-\delta(\tau-k)} \Phi(d_{1,k;K_2}) - K_2 e^{-r(\tau-k)} \Phi(d_{2,k;K_2}))^{m_2} f(S_k) dS_k \\
&= \int_0^{\infty} (e^x e^{-\delta(\tau-k)} \Phi(d_{1,k;K_1}) - K_1 e^{-r(\tau-k)} \Phi(d_{2,k;K_1}))^{m_1} \\
&\qquad (e^x e^{-\delta(\tau-k)} \Phi(d_{1,k;K_2}) - K_2 e^{-r(\tau-k)} \Phi(d_{2,k;K_2}))^{m_2} \frac{1}{\sqrt{2\pi k \sigma_k^2}} e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}} dx \\
&= \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \binom{m_1}{i} \binom{m_2}{j} e^{-(i+j)\delta(\tau-k)} (-K_1)^{m_1-i} (-K_2)^{m_2-j} e^{-r(\tau-k)(m_1+m_2-i-j)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{-\infty}^{\infty} e^{(i+j)x}\Phi(d_{1,k;K_1})^i\Phi(d_{2,k;K_1})^{m_1-i}\Phi(d_{1,k;K_2})^j\Phi(d_{2,k;K_2})^{m_2-j}\frac{1}{\sqrt{2\pi k\sigma_k^2}}e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}}dx \\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}e^{-(i+j)\delta(\tau-k)}(-K_1)^{m_1-i}(-K_2)^{m_2-j}e^{-r(\tau-k)(m_1+m_2-i-j)}e^{((i+j)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(i+j)^2\sigma_k^2}{2})k} \\
&\times \int_{-\infty}^{\infty} \Phi(d_{1,k;K_1})^i\Phi(d_{2,k;K_1})^{m_1-i}\Phi(d_{1,k;K_2})^j\Phi(d_{2,k;K_2})^{m_2-j}\frac{1}{\sqrt{2\pi k\sigma_k^2}}e^{-\frac{(x-(\mu-\delta+(i+j-\frac{1}{2})\sigma_k^2)k)^2}{2k\sigma_k^2}}dx \\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}e^{-(i+j)\delta(\tau-k)}(-K_1)^{m_1-i}(-K_2)^{m_2-j}e^{-r(\tau-k)(m_1+m_2-i-j)}e^{((i+j)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(i+j)^2\sigma_k^2}{2})k} \\
&\times \int_{-\infty}^{\infty} \Phi(A_1(i,j;K_1)+Bz)^i\Phi(A_2(i,j;K_1)+Bz)^{m_1-i} \\
&\times \Phi(A_1(i,j;K_2)+Bz)^j\Phi(A_2(i,j;K_2)+Bz)^{m_2-j}\phi(z)dz,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(i,j;K) &= \frac{(\mu-\delta+(i+j-\frac{1}{2})\sigma_k^2)k-\log(K)+(\tau-k)(r-\delta)+\frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma}, \\
A_2(i,j;K) &= A_1(i,j;K)-\sqrt{\tau-k}\sigma, \\
B &= \frac{\sqrt{k}\sigma_k}{\sqrt{\tau-k}\sigma}.
\end{aligned}
$$

Next, we compute the cross-moment of two put options with different strike prices K_1 and K_2

$$
\begin{aligned}
&E_t[P_k(K_1)^{m_1}P_k(K_2)^{m_2}] \\
&= \int_0^{\infty}(K_1e^{-r(\tau-k)}\Phi(-d_{2,k;K_1})-S_ke^{-\delta(\tau-k)}\Phi(-d_{1,k;K_1}))^{m_1} \\
&\qquad (K_2e^{-r(\tau-k)}\Phi(-d_{2,k;K_2})-S_ke^{-\delta(\tau-k)}\Phi(-d_{1,k;K_2}))^{m_2}f(S_k)dS_k \\
&= \int_0^{\infty}(K_1e^{-r(\tau-k)}\Phi(-d_{2,k;K_1})-e^xe^{-\delta(\tau-k)}\Phi(-d_{1,k;K_1}))^{m_1} \\
&\qquad (K_2e^{-r(\tau-k)}\Phi(-d_{2,k;K_2})-e^xe^{-\delta(\tau-k)}\Phi(-d_{1,k;K_2}))^{m_2}\frac{1}{\sqrt{2\pi k\sigma_k^2}}e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}}dx \\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}K_1^iK_2^je^{-r(\tau-k)(i+j)}(-1)^{m_1+m_2-i-j}e^{-(m_1+m_2-i-j)\delta(\tau-k)} \\
&\times \int_{-\infty}^{\infty} e^{(m_1+m_2-i-j)x}\Phi(-d_{2,k;K_1})^i\Phi(-d_{1,k;K_1})^{m_1-i}\Phi(-d_{2,k;K_2})^j\Phi(-d_{1,k;K_2})^{m_2-j}\frac{1}{\sqrt{2\pi k\sigma_k^2}}e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}K_1^iK_2^je^{-r(\tau-k)(i+j)}(-1)^{m_1+m_2-i-j}e^{-(m_1+m_2-i-j)\delta(\tau-k)}e^{((m_1+m_2-i-j)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m_1+m_2-i-j)^2\sigma_k^2}{2})k}\\
&\qquad\times\int_{-\infty}^{\infty}\Phi(-d_{2,k;K_1})^i\Phi(-d_{1,k;K_1})^{m_1-i}\Phi(-d_{2,k;K_2})^j\Phi(-d_{1,k;K_2})^{m_2-j}\frac{1}{\sqrt{2\pi k\sigma_k^2}}e^{-\frac{(x-(\mu-\delta+(m_1+m_2-i-j-\frac{1}{2})\sigma_k^2)k)^2}{2k\sigma_k^2}}dx\\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}K_1^iK_2^je^{-r(\tau-k)(i+j)}(-1)^{m_1+m_2-i-j}e^{-(m_1+m_2-i-j)\delta(\tau-k)}e^{((m_1+m_2-i-j)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(m_1+m_2-i-j)^2\sigma_k^2}{2})k}\\
&\qquad\times\int_{-\infty}^{\infty}\Phi(A_1(i,j;K_1)+Bz)^i\Phi(A_2(i,j;K_1)+Bz)^{m_1-i}\\
&\qquad\times\Phi(A_1(i,j;K_2)+Bz)^j\Phi(A_2(i,j;K_2)+Bz)^{m_2-j}\phi(z)dz,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(i,j;K) &= -\frac{(\mu-\delta+(m_1+m_2-i-j-\frac{1}{2})\sigma_k^2)k-\log(K)+(\tau-k)(r-\delta-\sigma^2)+\frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma},\\
A_2(i,j;K) &= A_1(i,j;K)-\sqrt{\tau-k}\sigma,\\
B &= -\frac{\sqrt{k}\sigma_k}{\sqrt{\tau-k}\sigma}.
\end{aligned}
$$

Lastly, we compute the cross-moment of call and put options with different strike prices K_1 and K_2

$$
\begin{aligned}
&E_t[C_k(K_1)^{m_1}P_k(K_2)^{m_2}]\\
&= \int_0^{\infty}(S_ke^{-\delta(\tau-k)}\Phi(d_{1,k;K_1})-K_1e^{-r(\tau-k)}\Phi(d_{2,k;K_1}))^{m_1}\\
&\qquad(K_2e^{-r(\tau-k)}\Phi(-d_{2,k;K_2})-S_ke^{-\delta(\tau-k)}\Phi(-d_{1,k;K_2}))^{m_2}f(S_k)dS_k\\
&= \int_0^{\infty}(e^xe^{-\delta(\tau-k)}\Phi(d_{1,k;K_1})-K_1e^{-r(\tau-k)}\Phi(d_{2,k;K_1}))^{m_1}\\
&\qquad(K_2e^{-r(\tau-k)}\Phi(-d_{2,k;K_2})-e^xe^{-\delta(\tau-k)}\Phi(-d_{1,k;K_2}))^{m_2}\frac{1}{\sqrt{2\pi k\sigma_k^2}}e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}}dx\\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}e^{-(i+m_2-j)\delta(\tau-k)}(-1)^{m_1+m_2-i-j}K_1^{m_1-i}K_2^je^{-r(\tau-k)(m_1-i+j)}\\
&\times\int_{-\infty}^{\infty}e^{(i+m_2-j)x}\Phi(d_{1,k;K_1})^i\Phi(d_{2,k;K_1})^{m_1-i}\Phi(-d_{2,k;K_2})^j\Phi(-d_{1,k;K_2})^{m_2-j}\frac{1}{\sqrt{2\pi k\sigma_k^2}}e^{-\frac{(x-(\mu-\delta-\frac{\sigma_k^2}{2})k)^2}{2k\sigma_k^2}}dx\\
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}e^{-(i+m_2-j)\delta(\tau-k)}(-1)^{m_1+m_2-i-j}K_1^{m_1-i}K_2^je^{-r(\tau-k)(m_1-i+j)}e^{((i+m_2-j)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(i+m_2-j)^2\sigma_k^2}{2})k}
\end{aligned}
$$

$$\times \int_{-\infty}^{\infty} \Phi(d_{1,k;K_1})^i \Phi(d_{2,k;K_1})^{m_1-i} \Phi(-d_{2,k;K_2})^j \Phi(-d_{1,k;K_2})^{m_2-j} \frac{1}{\sqrt{2\pi k \sigma_k^2}} e^{-\frac{(x-(\mu-\delta+(i+m_2-j-\frac{1}{2})\sigma_k^2)k)^2}{2k\sigma_k^2}} dx$$

$$= \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \binom{m_1}{i} \binom{m_2}{j} e^{-(i+m_2-j)\delta(\tau-k)} (-1)^{m_1+m_2-i-j} K_1^{m_1-i} K_2^j e^{-r(\tau-k)(m_1-i+j)} e^{((i+m_2-j)(\mu-\delta-\frac{\sigma_k^2}{2})+\frac{(i+m_2-j)^2\sigma_k^2}{2})k}$$

$$\times \int_{-\infty}^{\infty} \Phi(A_1(i,j;K_1) + Bz)^i \Phi(A_2(i,j;K_1) + Bz)^{m_1-i}$$

$$\times \Phi(A_1(i,j;K_2) - Bz)^j \Phi(A_2(i,j;K_2) - Bz)^{m_2-j} \phi(z) dz,$$

where

$$A_1(i,j;K_1) = \frac{(\mu-\delta+(i+m_2-j-\frac{1}{2})\sigma_k^2)k - \log(K_1) + (\tau-k)(r-\delta) + \frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma},$$

$$A_2(i,j;K_1) = A_1(i,j;K_1) - \sqrt{\tau-k}\sigma,$$

$$A_1(i,j;K_2) = -\frac{(\mu-\delta+(i+m_2-j-\frac{1}{2})\sigma_k^2)k - \log(K_2) + (\tau-k)(r-\delta-\sigma^2) + \frac{\tau-k}{2}\sigma^2}{\sqrt{\tau-k}\sigma},$$

$$A_2(i,j;K_2) = A_1(i,j;K_2) - \sqrt{\tau-k}\sigma,$$

$$B = \frac{\sqrt{k}\sigma_k}{\sqrt{\tau-k}\sigma}.$$

We can now compute the CAPM-beta of an option position held for k-period with strike price K as

$$\begin{aligned}
\beta(O_k(K)) &= \frac{\mathrm{Cov}\left(\frac{O_k(K)}{O_0(K)} - 1, \tilde{S}_k - 1\right)}{\mathrm{Var}(\tilde{S}_k - 1)} \\
&= \frac{1}{O_0(K)} \frac{\mathrm{Cov}(O_k(K), \tilde{S}_k)}{\mathrm{Var}(\tilde{S}_k)} \\
&= \frac{1}{O_0(K)} \frac{1}{(e^{k\sigma_k^2}-1)e^{2k(\mu-\frac{\sigma_k^2}{2})+k\sigma_k^2}} \left(E[\tilde{S}_k O_k(K)] - E[O_k(K)]E[\tilde{S}_k]\right) \\
&= \frac{1}{O_0(K)} \frac{1}{(e^{k\sigma_k^2}-1)e^{2k(\mu-\frac{\sigma_k^2}{2})+k\sigma_k^2}} \left(E[\tilde{S}_k O_k(K)] - e^{k\mu} E[O_k(K)]\right)
\end{aligned}$$

where $E[\tilde{S}_k O_k(K)]$ and $E[O_k(K)]$ are given in the expression derived above. And we can compute CAPM-alpha of an option position as

$$\alpha(O_k(K)) = E\left[\frac{O_k(K)}{O_0(K)} - 1\right] - R_f - \beta(O_k(K))(E[\tilde{S}_k - 1] - R_f)$$

$$= \frac{1}{O_0(K)} E[O_k(K)] - 1 - R_f - \beta(O_k(K))(e^{k\mu} - 1 - R_f)$$

where we have assumed that underlying asset S_t is the market index.

OA.2 Moments of Future Value of a European Option (Holding Period=Time to Maturity)

Suppose $X \sim N(\mu, \sigma^2)$. Then $Y = \exp(X)$ has a log-normal distribution. It is straightforward to show that

$$E[Y^m] = \exp\left(m\mu + \frac{m^2\sigma^2}{2}\right). \tag{1}$$

The lower, upper and doubly truncated moments of Y are given by

$$\int_a^\infty Y^m f(Y)\mathrm{d}Y = E[Y^m]\Phi(m\sigma - a_0), \tag{2}$$
$$\int_0^b Y^m f(Y)\mathrm{d}Y = E[Y^m]\Phi(b_0 - m\sigma), \tag{3}$$
$$\int_a^b Y^m f(Y)\mathrm{d}Y = E[Y^m][\Phi(m\sigma - a_0) - \Phi(m\sigma - b_0)], \tag{4}$$
$$\tag{5}$$

where

$$a_0 = \frac{\log(a) - \mu}{\sigma}, \tag{6}$$
$$b_0 = \frac{\log(b) - \mu}{\sigma}. \tag{7}$$

Suppose

$$\log(S_{t+\tau}) \sim N\left(\log(S_t) + \tau\left(\mu - \frac{\sigma^2}{2}\right), \tau\sigma_\tau^2\right). \tag{8}$$

From the above result, we have

$$E[C_{t+\tau}^m] = \int_K^\infty (S_{t+\tau} - K)^m f(S_{t+\tau})\mathrm{d}S_{t+\tau}$$
$$= \sum_{i=0}^m \binom{m}{i}(-K)^{m-i}\int_K^\infty S_{t+\tau}^i f(S_{t+\tau})\mathrm{d}S_{t+\tau}$$

$$= \sum_{i=0}^{m} \binom{m}{i} (-K)^{m-i} E[S_{t+\tau}^i] \Phi(i\sqrt{\tau}\sigma_\tau - \tilde{K}), \tag{9}$$

where

$$\tilde{K} = \frac{\log(K) - \log(S_t) - \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right)}{\sqrt{\tau}\sigma_\tau} = -\frac{\log\left(\frac{S_t}{K}\right) + \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right)}{\sqrt{\tau}\sigma_\tau}, \tag{10}$$

$$E[S_{t+\tau}^i] = S_t^i \exp\left(i\tau\left(\mu - \frac{\sigma_\tau^2}{2}\right) + \frac{i^2\tau\sigma_\tau^2}{2}\right). \tag{11}$$

Similarly, we have

$$\begin{aligned} E[P_{t+\tau}^m] &= \int_0^K (K - S_{t+\tau})^m f(S_{t+\tau}) \mathrm{d}S_{t+\tau} \\ &= \sum_{i=0}^{m} \binom{m}{i} K^{m-i} (-1)^i \int_0^K S_{t+\tau}^i f(S_{t+\tau}) \mathrm{d}S_{t+\tau} \\ &= \sum_{i=0}^{m} \binom{m}{i} (K)^{m-i} (-1)^i E[S_{t+\tau}^i] \Phi(\tilde{K} - i\sqrt{\tau}\sigma_\tau). \end{aligned} \tag{12}$$

For example, when $m = 1$, we have

$$E[C_{t+\tau}] = S_t e^{\tau\mu} \Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right) + \tau\sigma_\tau^2}{\sqrt{\tau}\sigma_\tau}\right) - K\Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right)}{\sqrt{\tau}\sigma_\tau}\right), \tag{13}$$

$$E[P_{t+\tau}] = K\Phi\left(-\frac{\log\left(\frac{S_t}{K}\right) + \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right)}{\sqrt{\tau}\sigma_\tau}\right) - S_t e^{\tau\mu} \Phi\left(-\frac{\log\left(\frac{S_t}{K}\right) + \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right) + \tau\sigma_\tau^2}{\sqrt{\tau}\sigma_\tau}\right). \tag{14}$$

After simplification, we can show that

$$E[C_{t+\tau}] = e^{r\tau} C^{BS}(\tilde{S}_t, K, r, \tau, \sigma_\tau), \tag{15}$$

$$E[P_{t+\tau}] = e^{r\tau} P^{BS}(\tilde{S}_t, K, r, \tau, \sigma_\tau), \tag{16}$$

where[7]

$$\tilde{S}_t = S_t \exp\left((\mu - r)\tau\right). \tag{17}$$

[7]Note that although the option pricing formula depends on the interest rate r, $E[C_{t+\tau}]$ and $E[P_{t+\tau}]$ (as well as other mixed moments) are independent of r.

When $m = 2$, we have

$$E[C_{t+\tau}^2] = K^2\Phi(-\tilde{K}) - 2KE[S_{t+\tau}]\Phi(\sqrt{\tau}\sigma_\tau - \tilde{K}) + E[S_{t+\tau}^2]\Phi(2\sqrt{\tau}\sigma_\tau - \tilde{K}), \tag{18}$$

$$E[P_{t+\tau}^2] = K^2\Phi(\tilde{K}) - 2KE[S_{t+\tau}]\Phi(\tilde{K} - \sqrt{\tau}\sigma_\tau) + E[S_{t+\tau}^2]\Phi(\tilde{K} - 2\sqrt{\tau}\sigma_\tau). \tag{19}$$

The same approach also allows us to compute mixed moments of payoffs of different calls and puts. Suppose we have two calls with exercise price $K_1 > K_2$. We can then compute the mixed moments of their payoffs as (with $m_1 > 0$ and $m_2 > 0$)

$$\begin{aligned}
&E[C_{t+\tau}(K_1)^{m_1} C_{t+\tau}(K_2)^{m_2}] \\
&= \int_{K_1}^{\infty} (S_{t+\tau} - K_1)^{m_1} (S_{t+\tau} - K_2)^{m_2} f(S_{t+\tau}) \mathrm{d}S_{t+\tau} \\
&= \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \binom{m_1}{i} \binom{m_2}{i} (-K_1)^{m_1-i} (-K_2)^{m_2-i} E[S_{t+\tau}^{i+j}] \Phi((i+j)\sqrt{\tau}\sigma_\tau - \tilde{K}_1),
\end{aligned} \tag{20}$$

where

$$\tilde{K}_1 = -\frac{\log\left(\frac{S_t}{K_1}\right) + \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right)}{\sqrt{\tau}\sigma_\tau}. \tag{21}$$

For $K_1 > K_2$, the mixed moments of payoffs of call and put are given by

$$E[C_{t+\tau}(K_1)^{m_1} P_{t+\tau}(K_2)^{m_2}] = 0. \tag{22}$$

A particular interesting case is the mixed moments of the payoffs of a call and a put with the same exercise price are always equal to zero. This is because when one has non-zero payoff, the other has zero payoff, and vice versa. In particular, we have

$$\begin{aligned}
\mathrm{Cov}[C_{t+\tau}(K) P_{t+\tau}(K)] &= E[C_{t+\tau}(K) P_{t+\tau}(K)] - E[C_{t+\tau}(K)] E[P_{t+\tau}(K)] \\
&= -E[C_{t+\tau}(K)] E[P_{t+\tau}(K)].
\end{aligned} \tag{23}$$

However, the other mixed moments are nonzero and they are given by

$$\begin{aligned}
&E[P_{t+\tau}(K_1)^{m_1} C_{t+\tau}(K_2)^{m_2}] \\
&= \int_{K_2}^{K_1} (K_1 - S_{t+\tau})^{m_1} (S_{t+\tau} - K_2)^{m_2} f(S_{t+\tau}) \mathrm{d}S_{t+\tau} \\
&= \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \binom{m_1}{i} \binom{m_2}{i} K_1^{m_1-i} (-K_2)^{m_2-j} (-1)^i \int_{K_2}^{K_1} S_{t+\tau}^{i+j} f(S_{t+\tau}) \mathrm{d}S_{t+\tau}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{i}K_1^{m_1-i}(-K_2)^{m_2-j}(-1)^i E[S_{t+\tau}^{i+j}] \\
&\quad \times \left[\Phi((i+j)\sqrt{\tau}\sigma_\tau - \tilde{K}_2) - \Phi((i+j)\sqrt{\tau}\sigma_\tau - \tilde{K}_1)\right], \qquad (24)
\end{aligned}
$$

where

$$
\tilde{K}_2 = -\frac{\log\left(\frac{S_t}{K_2}\right) + \tau\left(\mu - \frac{\sigma_\tau^2}{2}\right)}{\sqrt{\tau}\sigma_\tau}. \qquad (25)
$$

OA.3 Mean-Variance Under the Black-Scholes

Assume that stock price follows the Geometric Brownian motion as in the Black-Scholes model.

$$
dS_t = \mu S_t dt + \sigma S_t dW_t
$$

We are interested in the following optimization problem where the objective function is mean-variance utility.

$$
\begin{aligned}
\max_{w_o, w_s} \quad & E_t[r_{t:T}^M] - \frac{\gamma}{2}\text{Var}_t[r_{t:T}^M] \\
= \quad & E_t[R_{t:T}^M - 1] - \frac{\gamma}{2}\text{Var}_t[R_{t:T}^M - 1] \\
= \quad & E_t\left[\left(w_o\frac{O_T}{O_t} + w_s\frac{S_T}{S_t} + (1 - w_o - w_s)(1 + r_t)\right) - 1\right] \\
& -\frac{\gamma}{2}\text{Var}_t\left[\left(w_o\frac{O_T}{O_t} + w_s\frac{S_T}{S_t} + (1 - w_o - w_s)(1 + r_t)\right) - 1\Big)\right]
\end{aligned}
$$

OA.4 Mean-Variance Portfolio Analysis with Solvency Constraints

We discusses how to perform mean-variance analysis of a portfolio with a risk-free stock, a stock, and a number of call options on the stock, when the portfolio is required to satisfy the solvency constraints, i.e., the terminal value of the portfolio has to be nonnegative.

Let $w = [w_1, \ldots, w_{n+1}]'$ be the weights on the stock and n call options on the stock, with exercise prices $K_1 < K_2 < \cdots < K_n$. The weight on the risk-free asset is

$$w_0 = 1 - \sum_{i=1}^{n+1} w_i. \tag{1}$$

Let μ and Σ be the expected return and the covariance matrix of the stock and the n calls. We consider an investor with a risk aversion coefficient of γ and a mean-variance utility of

$$U(w) = w'\mu - \frac{\gamma}{2} w'\Sigma w. \tag{2}$$

The unconstrained problem is easy to solve, it is given by

$$w^* = \mathrm{argmax}_w U(w) = \frac{1}{\gamma}\Sigma^{-1}\mu. \tag{3}$$

However, the terminal value of the resulting portfolio could be negative.

Let k be the holding period of the portfolio and S_k be the price of the stock at $t = k$. For a given weight w, the value of the portfolio at $t = k$ is

$$V_k = w_0 e^{rk} + w_1 S_k + \sum_{i=1}^{n} \frac{w_{i+1}}{C_0(K_i)} C_k(K_i), \tag{4}$$

where $C_t(K_i)$ is the value of the call with exercise price K_i at time t. We would like to find the optimal portfolio with the constraint that $V_k \geq 0$.

There are two cases to consider. The first case is the maturity of the call option, T, is the same as the holding period k. The second case is the maturity of the call option is longer than the holding period, i.e., $T > k$. For the first case, we have

$$C_k(K_i) = \max[S_k - K_i, 0]. \tag{5}$$

For the second case, we have

$$C_k(K_i) = C(S_k, T-k, K_i), \tag{6}$$

where $C(S, \tau, K)$ is the Black-Scholes price of a call with exercise price K, maturity τ when the price of the stock is S.

Case 1: $T = k$

When $T = k$, V_k is a piecewise linear function of S_k. In particular, we have

$$V_k = w_0 e^{rk} + w_1 S_k + \sum_{i=1}^{n} \frac{w_{i+1}}{C_0(K_i)} \max[S_k - K_i, 0]. \tag{7}$$

As a result, we just need to make sure $V_k \geq 0$ at $S_k = 0$ and $S_k = K_i$ for $i = 1, \ldots, n$. In addition, we need to make sure that when $S_k > K_n$ V_k is a nonnegative function of S_k. This implies the following $n + 2$ inequalities

$$w_0 \geq 0, \tag{8}$$

$$w_0 + w_1 K_i e^{-rk} + \sum_{j=1}^{i-1} \frac{w_{j+1}}{C_0(K_j)} (K_i - K_j) e^{-rk} \geq 0, \quad i = 1, \ldots, n, \tag{9}$$

$$w_1 + \sum_{j=1}^{n} \frac{w_{j+1}}{C_0(K_j)} \geq 0. \tag{10}$$

Note that when $n = 0$ (i.e., no call options), we have two constraints $w_0 \geq 0$ and $w_1 \geq 0$, which implies no short-selling and no borrowing. Using (1), we can write the above inequalities as $Aw \leq b$, where

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 - K_1 e^{-rk} & 1 & \cdots & 1 \\ 1 - K_2 e^{-rk} & 1 - \frac{(K_2 - K_1)e^{-rk}}{C_0(K_1)} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 - K_n e^{-rk} & 1 - \frac{(K_n - K_1)e^{-rk}}{C_0(K_1)} & \cdots & 1 - \sum_{j=1}^{n-1} \frac{(K_n - K_j)e^{-rk}}{C_0(K_j)} \\ -1 & -\frac{1}{C_0(K_1)} & \cdots & -\frac{1}{C_0(K_n)} \end{bmatrix}, \tag{11}$$

$$b = \begin{bmatrix} 1_{n+1} \\ 0 \end{bmatrix}. \tag{12}$$

Case 2: $T > k$

When $T > k$, V_k is a continuous function of S_k. For a given weight vector w, the solvency constraint requires $V_k \geq 0$ for $0 \leq S_k < \infty$. When $S_k = 0$, we have $V_k = w_0 e^{rk}$, so we need $w_0 \geq 0$ as in Case 1. In addition, we need $\partial V_k / \partial S_k \geq 0$ as $S_k \to \infty$. Note that

$$\frac{\partial V_k}{\partial S_k} = w_1 + \sum_{j=1}^{n} \frac{w_{j+1}}{C_0(K_j)} \Phi(d_{1,j}), \tag{13}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function and

$$d_{1,j} = \frac{\ln(S_k/K_j) + \left(r + \frac{\sigma^2}{2}\right)(T-k)}{\sqrt{T-k}\sigma}. \tag{14}$$

As $\lim_{S_k \to \infty} \Phi(d_{1,j}) = 1$, we need the weights to satisfy

$$w_1 + \sum_{j=1}^{n} \frac{w_{j+1}}{C_0(K_j)} \geq 0. \tag{15}$$

This is also the same linear inequality as in Case 1.

When $0 < S_k < \infty$, we need

$$w_0 + w_1 S_k e^{-rk} + \sum_{j=1}^{n} \frac{w_{j+1}}{C_0(K_j)} \left[S_k e^{-rk}\Phi(d_{1,j}) - K_j e^{-rT}\Phi(d_{2,j})\right] \geq 0, \tag{16}$$

where

$$d_{2,j} = d_{1,j} - \sqrt{T-k}\sigma. \tag{17}$$

Note that (16) is still a linear constraint on w except that the coefficients change with S_k.

We first study the special case of $n = 1$. In this case, we set $K = K_1$ and we need

$$w_0 + w_1 S_k e^{-rk} + \frac{w_2}{C_0(K)} e^{-rk} C(S_k, T-k, K) \geq 0. \tag{18}$$

Note that if $w_1 \geq 0$ and $w_2 \geq 0$, this inequality obviously holds, and if $w_1 < 0$ and $w_2 < 0$, this inequality is violated when S_k is large enough. It remains to consider (i) $w_1 \geq 0$ and $w_2 < 0$, and (ii) $w_1 \leq 0$ and $w_2 > 0$.

When $w_1 \geq 0$ and $w_2 \leq 0$ and if the following two inequalities hold

$$w_0 \geq 0, \tag{19}$$

$$w_1 + \frac{w_2}{C_0(K)} \geq 0, \tag{20}$$

we have

$$V_k = w_0 e^{rk} + w_1 S_k + \frac{w_2}{C_0(K)} C(S_k, T-k, K) \geq w_0 e^{rk} + w_1 S_k + \frac{w_2}{C_0(K)} S_k \geq 0, \tag{21}$$

so the solvency constraints are satisfied for $0 < S_k < \infty$.

We now focus on the case that $w_1 < 0$ and $w_2 > 0$. The minimum of V_k occurs when S_k satisfies the following first order condition.

$$\frac{\partial V_k}{\partial S_k} = w_1 + \frac{w_2}{C_0(K)}\Phi(d_1) = 0, \tag{22}$$

where

$$d_1 = \frac{\ln(S_k/K) + \left(r + \frac{\sigma^2}{2}\right)(T-k)}{\sqrt{T-k}\sigma}. \tag{23}$$

Note that the first order condition can be solved analytically. Let $\Phi^{-1}(\cdot)$ be the inverse cdf of a standard normal distribution, we obtain[8]

$$d_1^* = \Phi^{-1}\left(-\frac{w_1 C_0(K)}{w_2}\right). \tag{24}$$

With d_1^* available, we can easily obtain S_k^* that satisfies (22). It follows that the minimum value of V_k is

$$\begin{aligned} V_k^* &= w_0 e^{rk} + w_1 S_k^* + \frac{w_2}{C_0(K)}[S_k^*\Phi(d_1^*) - Ke^{-r(T-k)}\Phi(d_2^*)] \\ &= w_0 e^{rk} - \frac{w_2}{C_0(K)}Ke^{-r(T-k)}\Phi(d_2^*) \\ &= e^{rk}\left[1 - w_1 - w_2 - \frac{w_2}{C_0(K)}Ke^{-rT}\Phi(d_2^*)\right], \end{aligned} \tag{25}$$

where the second equality holds because of the first order condition. We just need to impose $V_k^* \geq 0$ as the additional nonlinear constraint on w.

In summary, we first solve the quadratic programming problem with the two linear inequalities constraints on w. If the resulting $w_1 > 0$, then we know the solvency constraint is satisfied. If $w_1 \leq 0$, we need to solve the quadratic programming problem with the usual two linear constraints together with $w_1 \leq 0$ and the additional nonlinear constraint.

General Case

We now consider $n > 1$. The two linear constraints are

$$w_0 \geq 0, \tag{26}$$

$$w_1 + \sum_{j=1}^{n} \frac{w_{j+1}}{C_0(K_j)} \geq 0. \tag{27}$$

[8] Since we require $w_1 + w_2/C_0(K) \geq 0$, so $-w_1 C_0(K)/w_2 \leq 1$.

Claim: Let

$$Q_i = w_1 + \sum_{j=1}^{i} \frac{w_{j+1}}{C_0(K_j)}. \tag{28}$$

Suppose w_{I+1} is the last element of w_i that takes negative value. If $Q_i \geq 0$ for $i = 0, \ldots, I$, then $V_k \geq 0$ when (26) and (27) hold.

Proof: When $w_i \geq 0$ for $i > I + 1$ and $w_0 \geq 0$, we have

$$\begin{aligned} V_k &\geq w_1 S_k + \sum_{i=1}^{I} \frac{w_{i+1}}{C_0(K_j)} C_k(K_j) \\ &> \left[w_1 + \frac{w_2}{C_0(K_1)} \right] C_k(K_1) + \sum_{i=2}^{I} \frac{w_{i+1}}{C_0(K_i)} C_k(K_i) \\ &> \left[w_1 + \frac{w_2}{C_0(K_1)} + \frac{w_3}{C_0(K_2)} \right] C_k(K_2) + \sum_{i=3}^{I} \frac{w_{i+1}}{C_0(K_i)} C_k(K_i) \\ &> \left[w_1 + \sum_{i=1}^{I} \frac{w_{i+1}}{C_0(K_i)} \right] C_k(K_I) > 0. \end{aligned} \tag{29}$$

This completes the proof.

Note that if $w_i \geq 0$ for $i \leq J + 1$, and $w_i \leq 0$ for $i > J + 1$ (i.e., w_i change sign once), then we obviously have $Q_i \geq 0$ for $i = 0, \ldots, n$. However, even when w_i change sign more than once, it is possible that $Q_i \geq 0$ for $i = 0, \ldots, I$, so the condition in our claim is weaker than w_i change sign once or less.

If Q_i takes negative value for some $i \leq I$, then it is possible that there exists a solution to the first order condition

$$\frac{\partial V_k}{\partial S_k} = w_1 + \sum_{i=1}^{n} \frac{w_i}{C_0(K_i)} \Phi(d_{1,i}) = 0, \tag{30}$$

where

$$d_{1,i} = \frac{\ln(S_k/K_i) + \left(r + \frac{\sigma^2}{2}\right)(T-k)}{\sqrt{T-k}\sigma}. \tag{31}$$

Let S_k^* be the solution to (30).[9] When $S_k = S_k^*$, we have

$$V_k^* = w_0 e^{rk} + w_1 S_k^* + \sum_{i=1}^{n} \frac{w_{i+1}}{C_0(K_i)} C(S_k^*, T-k, K_i)$$

[9] It is possible that (22) has more than one solution.

$$
\begin{aligned}
&= w_0 e^{rk} - \sum_{i=1}^{n} \frac{w_{i+1}}{C_0(K_i)} K_i e^{-r(T-k)} \Phi(d_{2,i}^*) \\
&= e^{rk} \left[1 - \sum_{i=1}^{n+1} w_i - \sum_{i=1}^{n} \frac{w_{i+1}}{C_0(K_i)} K_i e^{-rT} \Phi(d_{2,i}^*) \right],
\end{aligned} \tag{32}
$$

where

$$
d_{2,i}^* = \frac{\ln(S_k^*/K_i) + \left(r + \frac{\sigma^2}{2}\right)(T-k)}{\sqrt{T-k}\sigma} - \sqrt{T-k}\sigma, \tag{33}
$$

and we have the following nonlinear constraint on w:

$$
1 - \sum_{i=1}^{n+1} w_i - \sum_{i=1}^{n} \frac{w_{i+1}}{C_0(K_i)} K_i e^{-rT} \Phi(d_{2,i}^*) > 0. \tag{34}
$$

In summary, we first solve the quadratic programming problem with the two linear constraints (26) and (27). If the resulting weights are such that $Q_i \geq 0$ for $i = 0, \ldots I$, the optimal solution satisfies the solvency constraint. If not, we need to solve the optimization problem by imposing the nonlinear constraint.

Solvency Constraints with Both Calls and Puts

We now consider the solvency constraints when the portfolio has both calls and puts. Let $I_i = 0$ if the i-th option is a call and $I_i = 1$ if the i-th option is a put. For the case that $T = k$, we need to ensure that $V_k \geq 0$ for $S_k = 0$, $S_k = K_i$ for $i = 1, \ldots, n$ as well as for $S_k \to \infty$. This implies

$$
w_0 e^{rk} + \sum_{\substack{j=1, \\ I_j=1}}^{n} \frac{w_{j+1}}{P_0(K_j)} K_j \geq 0, \tag{35}
$$

$$
w_0 e^{rk} + w_1 K_i + \sum_{\substack{j=i+1, \\ I_j=1}}^{n} \frac{w_{j+1}}{P_0(K_j)} (K_j - K_i) + \sum_{\substack{j=1, \\ I_j=0}}^{i-1} \frac{w_{j+1}}{C_0(K_j)} (K_i - K_j) \geq 0, \quad i = 1, \ldots, n, \tag{36}
$$

$$
w_1 + \sum_{\substack{j=1, \\ I_j=0}}^{n} \frac{w_{j+1}}{C_0(K_j)} \geq 0. \tag{37}
$$

When there are no solvency constraints, the optimal portfolio with calls and puts are identical because the put-call parity suggests

$$P_k(K_i) = C_k(K_i) - S_k + K_i, \tag{38}$$
$$P_0(K_i) = C_0(K_i) - S_0 + K_i e^{-rk}. \tag{39}$$

Suppose w_{i+1} is the weight on a put with exercise price K_i in the optimal portfolio. We can replace it with a call with the exercise price K_i but with weight

$$\tilde{w}_{i+1} = \frac{w_{i+1}}{P_0(K_i)} C_0(K_i). \tag{40}$$

In addition, the new weights on the bond and the stock are

$$\tilde{w}_0 = w_0 + \frac{w_{i+1}}{P_0(K_i)} K_i e^{-rk}, \tag{41}$$
$$\tilde{w}_1 = w_1 - \frac{w_{i+1}}{P_0(K_i)}. \tag{42}$$

Due to the put-call parity, this new portfolio has the same payoffs as the old portfolio, i.e., $\tilde{V}_k = V_k$. It is important to note that even with solvency constraints, the optimal portfolio with both calls and puts is still identical to the optimal portfolio with just calls. This is because $\tilde{V}_k = V_k$, so if $V_k > 0$, $\tilde{V}_k$ must also be positive.

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